An extension of the spacetime symmetry group and its relation to SUSY

based on `arXiv:1507.08039`, `arXiv:1512.03328`

András LÁSZLÓ
`laszlo.andras@wigner.mta.hu`
Wigner RCP, Budapest, Hungary

CERS6, Prague

28th January 2016

Outline

This talk is not about GR itself.

But about how GR symmetries could put constraints on particle field theory.

- Introduction
- Structure of Lie groups
- Extensions of the spacetime symmetry group
- Unification mechanism for gauge and spacetime symm.
- Summary

Introduction

- Larger symmetry group means simpler model:
  - Less possibilities for invariant Lagrangians
  - Relation between otherwise unrelated coupling factors.
  - etc etc.

- This motivates unification attempts of symmetries.

- An evident candidate:
  - Unification of internal (gauge) symmetry groups.
  - Relating internal (gauge) symmetries to spacetime symmetries.

- But to actually do this is not that simple. . .

- Coleman-Mandula no-go theorem [Phys.Rev.**159**(1967)1251]:
  unification of a compact gauge group with spacetime symmetries not possible in QFT.
  Only possibility is the trivial one: gauge group $\times$ Poincaré group.
- Similar, simpler theorem is of McGlinn [PRL**12**(1964)467].
- O'Raifeartaigh classification theorem of Poincaré gr. extensions [PR**139**(1965)B1052]:
  traditionally, this is also interpreted supporting the above.

- **Important:** gauge group assumed to have positive definite invariant scalar product
  $\Leftrightarrow$ gauge group is assumed to be $\mathrm{U}(1) \times \ldots \times \mathrm{U}(1) \times$ compact semisimple Lie group.
  - Group theoretical convenience: classification of semisimple groups well understood.
  - Experimental justification: Standard Model has $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$, which is such.
  - Field theory reason: necessary for positivity of energy.
    (Because in energy density of Yang-Mills field one has this invariant scalar product.)

- OK, it seems Poincaré group cannot be extended by a gauge group.
- But is there extension at all of the Poincaré Lie algebra in some *deformed* manner?
- Haag-Lopuszanski-Sohnius theorem [Nucl.Phys.**B88**(1975)257]:
  if we take a *graded* Lie algebra instead of Lie algebra, it is possible to extend.
  - This extension is the SUSY algebra.
  - The extended part is not (!) related to the gauge group, but something else.
  - At first glance, group theoretical meaning of *graded* Lie algebra is not clear.
    (Any set of continuous transformations will satisfy Lie group/algebra axioms,
    so how come we deform the Lie algebra axioms? What does that mean?)

- We show an extension of spacetime symmetry group, which is not the SUSY.
- It is extension in the sense of ordinary Lie group, not *graded* etc deformation.
- The extended part is really the gauge group, not something else.
- Price: the gauge group will be not compact semisimple.
  Only the "important" part of it (Levi factor) will be compact semisimple.
- This is just the condition for positivity of energy density! And no exotic particles needed.

Structure of Lie groups

Example: Poincaré group, i.e. the global symmetries of flat spacetime.

$$\text{Poincaré group} \quad = \quad \mathcal{T} \rtimes \mathcal{L}$$

- $\mathcal{L}$: homogeneous Lorentz group   ($\longleftarrow$ “important part, represented with nice matrices”).
- $\mathcal{T}$: group of spacetime translations   ($\longleftarrow$ “trivial part, a bit pathological”).

Levi decomposition theorem states that this is generic: every Lie group has such structure.

Levi decomposition theorem:

If E is a finite dimensional connected and simply connected Lie group, then one has

$$E = R \rtimes L,$$

where R is solvable (called the **radical**) and L is semisimple (called the **Levi factor**).

- Levi factor: “important part”, nondegenerate Killing form on Lie algebra.
- Radical: “pathological part”, degenerate directions of the Killing form.

- A look at the Levi factor L:

  L is **semisimple**: Killing form $(x, y) \mapsto \mathrm{Tr}(\mathrm{ad}_x \mathrm{ad}_y)$ is non-degenerate on its Lie algebra. Here, $\mathrm{ad}_x(\cdot) := [x, \cdot]$ for a Lie algebra element x. Killing form: is invariant inner product.

  Remember from a Yang-Mills theory Lagrangian?

  $$F(\nabla)_{ab} F(\nabla)^{ab} \quad \longleftarrow \quad (\text{this is basically the Killing form evaluated on } F_{ab}, F^{ab}).$$

- A look at the radical R:

  R is **solvable**: its Lie algebra is the degenerate directions of the Killing form. If r is the Lie algebra of R, then solvability $\Leftrightarrow$ $r^0 := r, \quad r^1 := [r^0, r^0], \quad r^2 := [r^1, r^1], \quad \ldots, \quad r^k := [r^{k-1}, r^{k-1}] = \{0\}$ for finite k.

  Special case: R is **nilpotent**, i.e. there is finite k such that $\mathrm{ad}_{x_1} \ldots \mathrm{ad}_{x_k} = 0$ for all $x_1, \ldots, x_k \in r$.

  Special case: R is **abelian**, i.e. for all $x \in r$ one has $\mathrm{ad}_x = 0$.

Example: (proper) Poincaré group, $\mathcal{P}$.

$$\mathcal{P} \quad = \quad \mathcal{T} \rtimes \mathcal{L}$$

- $\mathcal{T}$: group of spacetime translations (radical, abelian normal subgroup). Acts on the flat spacetime and fields over flat spacetime as

  $$x^a \quad \mapsto \quad x^a + d^a$$

  in terms of affine spacetime coordinates.
- $\mathcal{L}$: homogeneous Lorentz group (Levi factor).

Example: (proper) SUSY group [Nucl.Phys.**B76**(1974)477, Phys.Lett.**B51**(1974)239].

$$\text{SUSY group} \quad = \quad \mathcal{S} \rtimes \mathcal{L}$$

- $\mathcal{S}$: group of supertranslations (radical, nilpotent normal subgroup). Acts on the vector bundle superfields over the flat spacetime as

$$\begin{pmatrix} \theta^A \\ x^a \end{pmatrix} \mapsto \begin{pmatrix} \theta^A + \epsilon^A \\ x^a + d^a + \sigma^a_{AA'}\mathrm{i}\big(\theta^A\bar{\epsilon}^{A'} - \epsilon^A\bar{\theta}^{A'}\big) \end{pmatrix}$$

  in terms of "supercoordinates" (Grassmann valued two-spinor coordinates) and affine spacetime coordinates.
- $\mathcal{L}$: homogeneous Lorentz group (Levi factor).

Lie algebra of SUSY is often presented as *graded* Lie algebra, but this is not necessary.
Can also be viewed as ordinary Lie algebra/group as shown above. (Less confusing?)

Extensions of the spacetime symmetry group

O'Raifeartaigh classification theorem on Poincaré group extensions [PR**139**(1965)B1052]:

Let $E = R \rtimes L$ be connected and simply connected Lie group.
Assume that it contains the (covering of the proper) Poincaré group $\mathcal{P} = \mathcal{T} \rtimes \mathcal{L}$.

Then,

$$\text{either}: \quad \begin{array}{ccccc} E & = & R & \rtimes & L \\ \cup & & \cup & & \cup \\ \mathcal{P} & = & \mathcal{T} & \rtimes & \mathcal{L} \end{array} \qquad \text{or}: \quad \begin{array}{cccc} E & = & R \rtimes & L \\ \cup & & & \cup \\ \mathcal{P} & = & & (\mathcal{T} \rtimes \mathcal{L}) \end{array}$$

Corollary: Exactly one of the below cases must hold.

(i) $E = \{\text{some semisimple Lie group}\} \times \mathcal{P}$. (⟵ trivial extension, Coleman-Mandula)

(ii) R is an abelian extension of $\mathcal{T}$, and L contains $\mathcal{L}$. (⟵ possible extra translations)

(iii) R is non-abelian extension of $\mathcal{T}$, and L contains $\mathcal{L}$. (⟵ SUSY, and our mechanism)

(iv) L contains entire $\mathcal{P}$, and L is simple. (⟵ rather artificial, basically impossible)

SUSY group is an extension of the Poincaré group with extended radical:

$$\begin{array}{ccccc} \text{SUSY group} & = & \mathcal{S} & \rtimes & \mathcal{L} \\ \cup & & \cup & & \parallel \\ \text{Poincaré group} & = & \mathcal{T} & \rtimes & \mathcal{L} \end{array}$$

SUSY group does not act on matter fields pointwise:

$$\mathcal{T} \subset \mathcal{S} \qquad \text{is normal subgroup}$$

but

$$\mathcal{S} \neq \mathcal{T} \rtimes \{\text{some other subgroup}\}$$

thus

$$\text{SUSY group} \neq \mathcal{T} \rtimes \{\text{some group acting at points of spacetime}\}$$

Thus: SUSY group is a bit unusual as a locally acting group.

Also: extended part of SUSY is not directly related to gauge group.

Our Poincaré group extension will be, however, of the form:

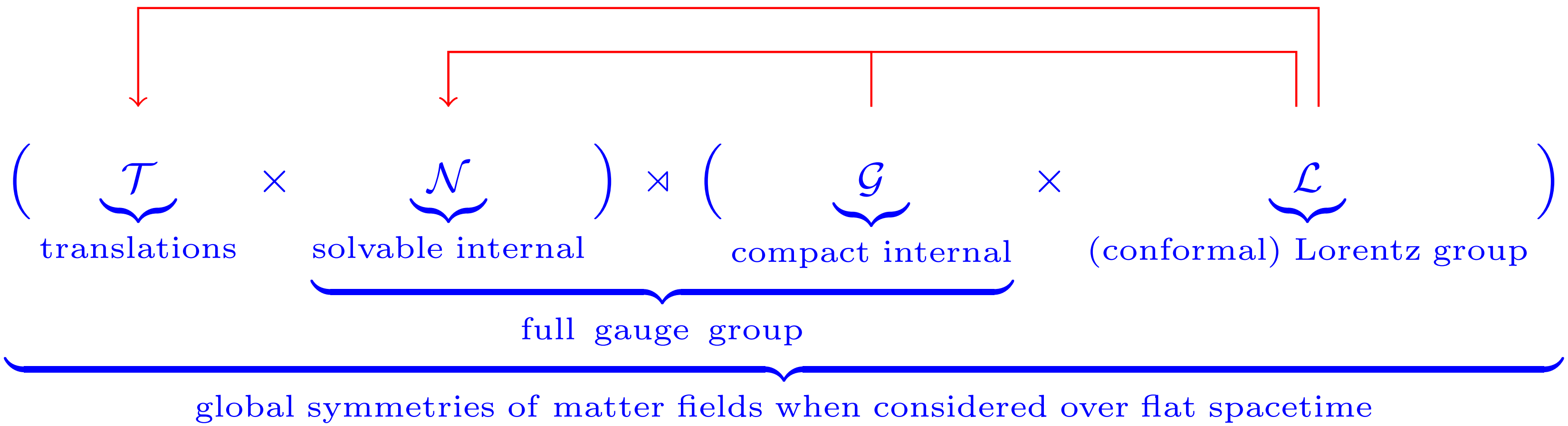
$$\mathcal{T} \rtimes \{\text{some group acting at points of spacetime}\}$$

$\Rightarrow$ can easily be transformed into a local (pointwise acting) group.

Also: the extended part shall directly be related to the gauge group.

Unification mechanism for gauge and spacetime symmetries

O'Raifeartaigh theorem allows following gauge - spacetime symmetry unification mechanism:



($\leftarrow$: illustrates direction of subgroup action over normal subgroups.)

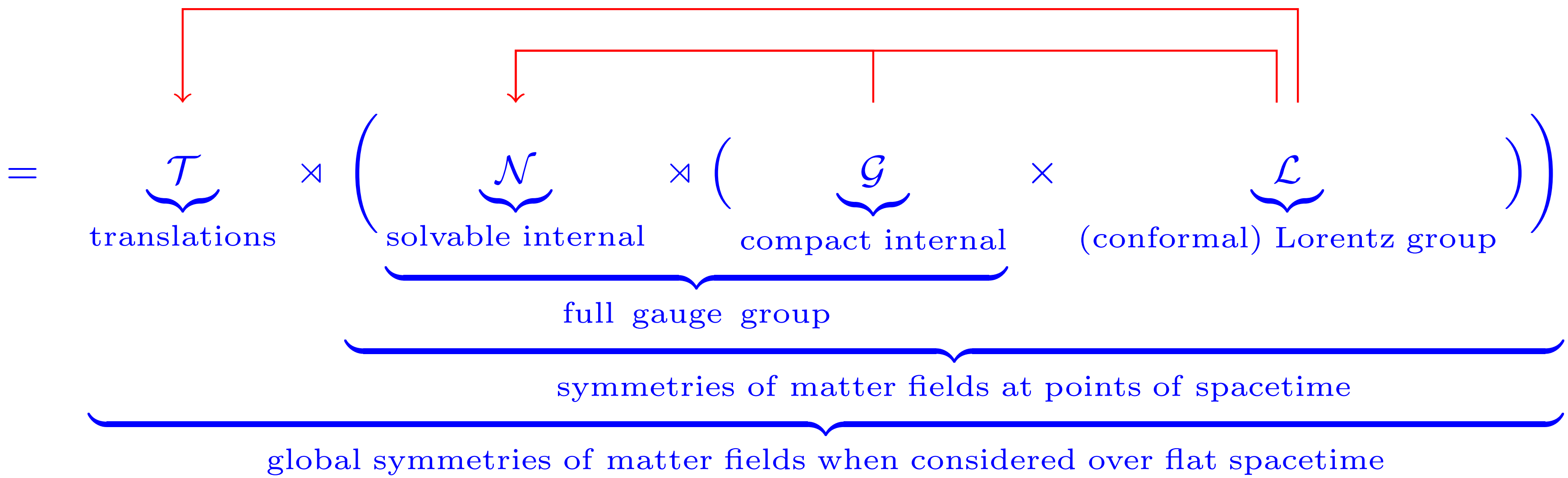
$\mathcal{G}$: usual compact gauge group. E.g. $\mathcal{G} = \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ in case of Standard Model.

$\mathcal{N}$: solvable extension of the gauge group. Glues together otherwise independent $\mathcal{G}$ and $\mathcal{L}$.

$\mathcal{N} \rtimes \mathcal{G}$: the full, slightly extended gauge group. (Price to pay.)

This is just the condition for energy positivity! (Killing form positive *semi*definite over $\mathcal{N} \rtimes \mathcal{G}$.)

The proposed group structure, using semi-associativity of $\rtimes$ and $\times$



Therefore, the non-translation part may be easily regarded as a locally acting group. (Acting on matter fields at points of spacetime.)

I.e. a group having the above structure may be easily “gauged” (made a local symmetry).

When considering this as a local symmetry, a unified local symmetry group of the form

$$\underbrace{\underbrace{\underbrace{\mathcal{N}}_{\text{solvable internal}} \rtimes \Big(\underbrace{\mathcal{G}}_{\text{compact internal}}}_{\text{full gauge group}} \times \underbrace{\mathcal{L}}_{\text{(conformal) Lorentz group}} \Big)}_{\text{symmetries of matter fields at points of spacetime}}$$

acts on matter fields at the points of spacetime.

$\mathcal{N}$ can act like a glue between compact gauge group $\mathcal{G}$ and spacetime symmetries $\mathcal{L}$.

If $\mathcal{N}$ not present: $\mathcal{G}$ and $\mathcal{L}$ can only be joined trivially, as dictated by Coleman-Mandula.

The unified local symmetry, using again semi-associativity of $\rtimes$ and $\times$

$$= \underbrace{\Big(\underbrace{\mathcal{N}}_{\text{solvable internal}} \rtimes \underbrace{\mathcal{G}}_{\text{compact internal}} \Big)}_{\text{full gauge group}} \rtimes \underbrace{\mathcal{L}}_{\text{(conformal) Lorentz group}}$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{symmetries of matter fields at points of spacetime}}$$

$\Rightarrow \mathcal{N} \rtimes \mathcal{G}$ is a normal subgroup complementing $\mathcal{L}$.

$\Rightarrow$ There is a homomorphism from the full group onto $\mathcal{L}$ whose kernel is $\mathcal{N} \rtimes \mathcal{G}$.

$\Rightarrow$ The full group has four-vector representation through $\mathcal{L}$.

$\Rightarrow$ The internal symmetries $\mathcal{N} \rtimes \mathcal{G}$ act trivially on four-vectors.

$\Rightarrow$ The four-vector representation acts as the conformal Lorentz group on four-vectors.

$\Leftrightarrow$ A Lorentz metric conformal equivalence class is associated to the full symmetry group.

$\Leftrightarrow$ A causal structure is associated to the full symmetry group.

Concrete example with $\mathcal{G} = \mathrm{U}(1)$ [`arXiv:1507.08039`, `arXiv:1512.03328`]:

the algebra automorphism group of the unital associative algebra $A \equiv \Lambda(\bar{S}^*) \otimes \Lambda(S^*)$.

S^*: dual two-spinor space.

$\bar{S}^*$: complex conjugate of dual two-spinor space.

$\Lambda(\cdot)$: exterior algebra.

QFT picture behind A:
creation operator algebra of a formal QFT system at a point of spacetime, containing a 2 internal degree of freedom Fermion particle, obeying Pauli principle, plus the antiparticle. (Can be related to CAR algebras.)

Important: $\mathrm{Aut}(A)$ contains a nilpotent normal subgroup N, the “dressing transformations”, not preserving the pure p, q-forms, i.e. pure p-fermions,q-antifermions ($\equiv \wedge^p \bar{S}^* \otimes \wedge^q S^*$)!

N mixes higher particle content to pure one-fermion states (hence the name).

Possible to show: $\mathrm{Aut}(A) \equiv N \rtimes \Big(\mathrm{U}(1) \times \mathcal{L}\Big)$.

A new mechanism for building unified theories?

Unification done at the same level as gravitation comes in.

Positivity of energy density of gauge fields satisfied.

No new exotic propagating gauge particles etc needed.

Gauge fields in the new sector have zero kinetic energy and zero kinetic Lagrangian.
$\Rightarrow$ Not real, physical propagating particles.
Good, because then they do not appear as yet unobserved exotic particles.
Their field equations are constraints associated to usual hyperbolic field equations.

Summary

- A non-SUSY Poincaré group extension was found.
- It is of the form $\{\text{translations}\} \rtimes \{\text{Lorentz group extension}\}$.
- The Lorentz group extension is an automorphism group.
- It has the structure
  $N \rtimes (\{\text{internal symmetries}\} \times \{\text{spacetime symmetries}\})$
  with nilpotent normal subgroup N.
- The internal and spacetime symmetries are glued by N.
- Coleman-Mandula theorem circuimvented because of N.
- Full gauge group $N \rtimes \{\text{internal symmetries}\}$
  has compact semisimple Levi factor
  $\Leftrightarrow$ positivity of energy. $+$ no new particles needed.
- Unification mechanism may be used for full SM group?

Backup slides

Let A be a finite dimensional complex associative algebra with unit, $\mathbb{1}$.

Let us have a conjugate-linear involution $(\cdot)^+ : A \to A$ on it such that for all $x, y \in A$ one has $(x\,y)^+ = x^+\,y^+$. We call then A as a $^+$**-algebra**.

(It is *not* a *-algebra, because $(x\,y)^+ \neq y^+\,x^+$.)

If A has a minimal generator system (e_1, e_2, e_3, e_4) such that

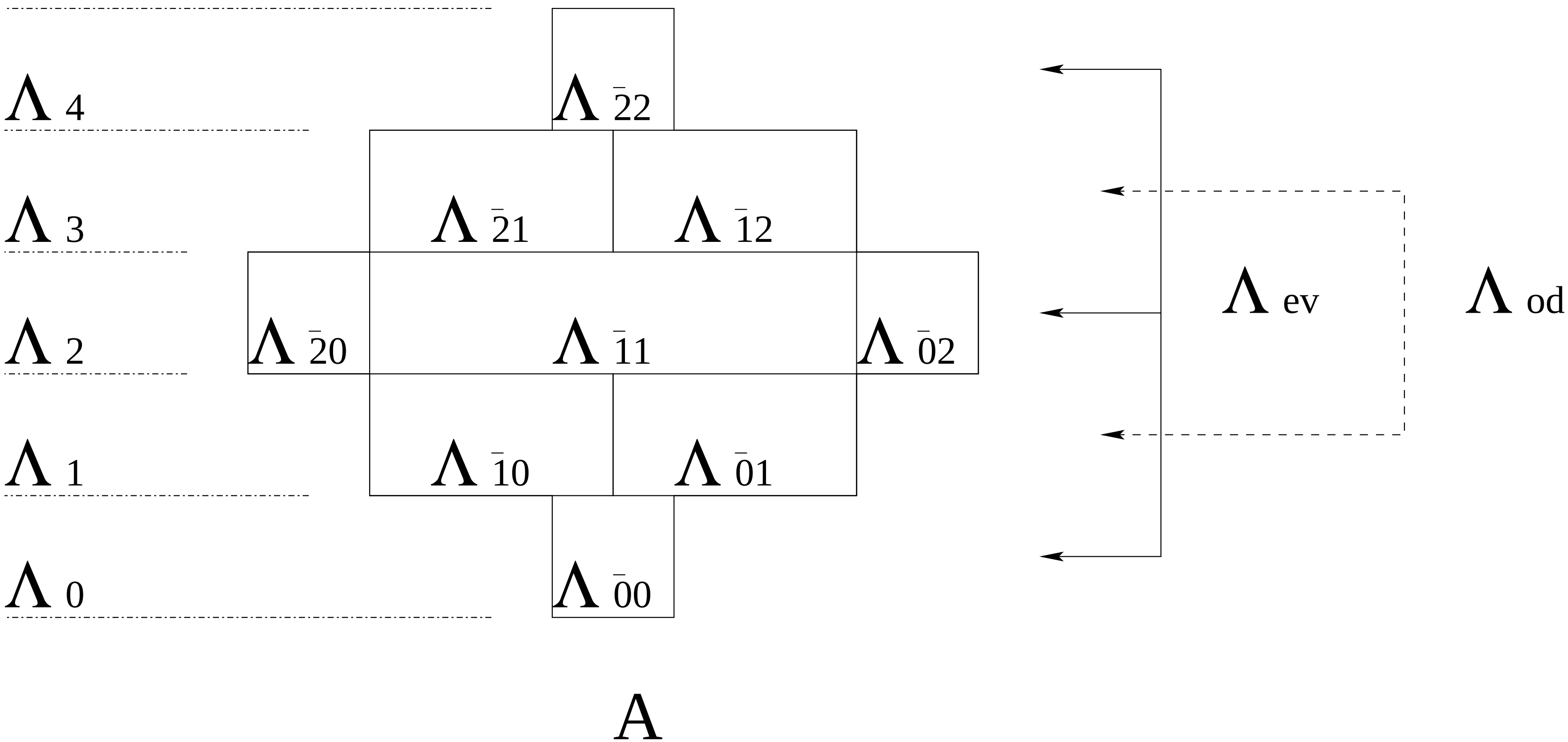
$$\begin{aligned} e_i e_j + e_j e_i &= 0 \quad (i,j \in \{1,2\} \text{ or } i,j \in \{3,4\}), \\ e_i e_j - e_j e_i &= 0 \quad (i \in \{1,2\} \text{ and } j \in \{3,4\}), \\ e_3 &= e_1^+, \\ e_4 &= e_2^+, \\ e_{i_1} e_{i_2} \dots e_{i_k} & \quad (1 \leq i_1 < i_2 < \dots < i_k \leq 4,\ 0 \leq k \leq 4) \\ & \quad \text{are linearly independent}. \end{aligned}$$

then we call it **spin algebra**,
the $^+$-operation as **charge conjugation**,
and a system of generators as above a **canonical generator system**.

Spin algebra $\sim$ algebra of creation operators of particle and antiparticle if we only had spin.

Illustration of spin algebra

(encoding 2 fundamental degrees of freedom, Pauli principle, and charge conjugation):



$\Lambda_{\bar{p}q}$:	p, q-forms, i.e. p-th polynomials of $\{e_1, e_2\}$ and q-th of $\{e_1^+, e_2^+\}$	($\mathbb{Z} \times \mathbb{Z}$-grading).
Λ_k:	k-forms, i.e. k-th polynomials of $\{e_1, e_2, e_1^+, e_2^+\}$	($\mathbb{Z}$-grading).
$\Lambda_{\text{ev}}, \Lambda_{\text{od}}$:	even/odd-forms, i.e. even/odd polynomials of $\{e_1, e_2, e_1^+, e_2^+\}$	($\mathbb{Z}_2$-grading).

Representation via two-spinor calculus:

Let S^* be a 2 dimensional complex vector space (“cospinor space”).

Then $\Lambda(\bar{S}^*) \otimes \Lambda(S^*)$ is spin algebra.

Here $\Lambda(some\ vector\ space)$ means exterior algebra of $some\ vector\ space$:
algebra of k-fold fully antisymmetric tensors of the vector space.

(Exterior algebra $\sim$ Grassmann algebra.)

So, a spin algebra is isomorphic to (“has the structure of”) $\Lambda(\bar{S}^*) \otimes \Lambda(S^*)$.
But there is a freedom in matching the canonical generators.

An element of $\Lambda(\bar{S}^*) \otimes \Lambda(S^*)$ consists of 9 spinorial sectors:

$$\Big(\varphi \quad \bar{\xi}_{(+)\,A'} \quad \xi_{(-)\,A} \quad \bar{\epsilon}_{(+)\,[A'B']} \quad v_{A'B} \quad \epsilon_{(-)\,[AB]} \quad \bar{\chi}_{(+)\,[C'D']A} \quad \chi_{(-)\,A'[CD]} \quad \omega_{[A'B'][CD]}\Big)$$

Our studied group:

$\mathrm{Aut}(A)$, the automorphism ("symmetry") group of spin algebra A.

These are the invertible $A \to A$ maps, which preserve the algebraic structure.

$\alpha \in \mathrm{Aut}(A) \Leftrightarrow$

- $\alpha : A \to A$ invertible transformation,
- α is complex-linear,
- α preserves algebraic product: $\alpha(x\,y) = \alpha(x)\,\alpha(y)$ for all $x, y \in A$,
- α is $^+$-real: $\alpha(x^+) = \alpha(x)^+$ for all $x \in A$.

Structure of $\mathrm{Aut}(A)$ (arXiv:1507.08039):

$$\mathrm{Aut}(A) = \underbrace{N}_{\text{grading non-preserving}} \rtimes \underbrace{\mathrm{Aut}_{\mathbb{Z}\times\mathbb{Z}}(A)}_{\mathbb{Z}\times\mathbb{Z}-\text{grading preserving}} \rtimes \underbrace{\mathcal{J}}_{\text{label exchanging}}$$

$\mathcal{J}$: particle-antiparticle label exchanging, i.e. $e_1 \mapsto e_3, \quad e_2 \mapsto e_4$ and vice-versa.

$\Lambda_{\bar{p}q} \leftrightarrow \Lambda_{\bar{q}p}$.

$\equiv \mathbb{Z}_2$ discrete group, trivial part.

$\mathrm{Aut}_{\mathbb{Z}\times\mathbb{Z}}(A)$: the $\mathbb{Z} \times \mathbb{Z}$-grading preserving automorphisms, i.e. preserving each $\Lambda_{\bar{p}q}$.

Just mixing e_1 and e_2 within each-other.

$\equiv \mathrm{GL}(2,\mathbb{C}) \equiv \mathrm{U}(1) \times \mathrm{D}(1) \times \mathrm{SL}(2,\mathbb{C})$ (here $\mathrm{D}(1)$: dilatation group).

N: nilpotent normal subgroup of **dressing transformations**.

Mixes higher form contribution to lower forms. Nontrivial part!

$e_i \mapsto e_i + \text{higher forms.}$

Structure of unit connected component of $\mathrm{Aut}(A)$ (omitting the trivial discrete part):

$$\underbrace{\underbrace{\underbrace{N}_{\text{dressing transformations}} \rtimes \Big(\underbrace{\mathrm{U}(1)}_{\text{internal symmetries}}}_{\text{full gauge group}} \times \underbrace{\mathrm{D}(1) \times \mathrm{SL}(2,\mathbb{C})}_{\text{spacetime symmetries}} \Big)}_{\text{full symmetry group of } A\text{-valued fields at a point of spacetime or mom. space}}$$

$\mathrm{D}(1) \times \mathrm{SL}(2,\mathbb{C})$: covering group of the homogeneous conformal Lorentz group.

$\mathrm{U}(1)$: can be regarded as a usual $\mathrm{U}(1)$ internal (gauge) symmetry.

N: nilpotent normal subgroup of dressing transformations, belonging to the gauge group.

N glues together the otherwise independent internal and spacetime symmetries.
(Coleman-Mandula theorem circuimvented.)

For Standard Model, one could search for something like:

$$\underbrace{\underbrace{N}_{\text{dressing transformations}} \rtimes \Big(\underbrace{\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)}_{\text{SM}-\text{internal symmetries}}}_{\text{full gauge group}} \times \underbrace{\mathrm{D}(1) \times \mathrm{SL}(2,\mathbb{C})}_{\text{spacetime symmetries}} \Big)}_{\text{full symmetry group of fields at a point of spacetime or momentum space}}$$

Full gauge group: $N \rtimes (\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3))$.

Allowed properties of the full gauge group:

- Has positive **semi**definite invariant inner product.
- Equivalently: only its **Levi factor** needs to be compact semisimple.

$\Rightarrow$ No problem with positivity of energy density, possible unification mechanism.