Existence theorem on the UV limit of Wilsonian renormalization group flows

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Outline

I. On Wilsonian RG flow of correlators (arbitrary signature):

- On manifolds: nice topological vector space behavior
- On flat spacetime for bosonic fields: $\exists$ of UV limit
- Is that true on manifolds?

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II. On Wilsonian RG flows of Feynman measures (Euclidean signature, flat spacetime, bosonic fields):

- $\exists$ of UV limit Feynman measure
- $\exists$ of UV limit interaction potential
- A new kind of Wilsonian renormalizability condition

[**`arXiv:2502.16319`**]

Part 0:

Notations, introduction

Distribution theory notations

Will consider only scalar valued fields for simplicity, see paper for vector valued case.

Will consider generic spacetime manifold, as well as flat spacetime case.

Usual distribution theory spaces:

only meaningful on flat spacetime

$$\underbrace{\mathcal{E}}_{\text{smooth fields}} \supset \underbrace{\mathcal{S}}_{\text{Schwartz fields}} \supset \underbrace{\mathcal{D}}_{\text{test fields}}$$

only meaningful on flat spacetime

$$\underbrace{\mathcal{E}'}_{\text{compactly supp distributions}} \subset \underbrace{\mathcal{S}'}_{\text{Schwartz distributions}} \subset \underbrace{\mathcal{D}'}_{\text{all distributions}}$$

Measure theory notations

(X, Σ, μ) is measure space iff:

- X is a set. [We only deal with $X = \mathcal{E}, \mathcal{S}, \mathcal{D}, \mathcal{E}', \mathcal{S}', \mathcal{D}'$.]
- Σ is a sigma-algebra of subsets of X.
  Usually, X carries topology $\rightarrow$ we take the Borel sigma-algebra. [Above X are Souslin.]
- $\mu : \Sigma \rightarrow \overline{\mathbb{R}}_0^+$ is a sigma-additive measure. Will only deal with finite measures.

Pushforward (or marginal) measure:

- Let (X, Σ, μ) be measure space and (Y, Δ) measurable space.
  Let $C : X \rightarrow Y$ be a measurable mapping.
- Pushforward (or marginal) measure $C_* \mu$ on Y is:

  for all $B \in \Delta$ one defines $(C_* \mu)(B) := \mu\big(\overset{-1}{C}(B)\big)$.

  One has $\int\limits_{\varphi \in \mathrm{Ran}(C)} f(\varphi)\, \mathrm{d}\big(C_* \mu\big)(\varphi) = \int\limits_{\phi \in X} f(C(\phi))\, \mathrm{d}\mu(\phi)$ for $f : \mathrm{Ran}(C) \rightarrow \overline{\mathbb{R}}$.

Fourier transform:

- Let μ be a finite measure e.g. on $X = \mathcal{E}, \mathcal{S}, \mathcal{D}, \mathcal{E}', \mathcal{S}', \mathcal{D}'$. Then $Z : X' \rightarrow \mathbb{C}$,
  $Z(j) := \int\limits_{\phi \in X} \mathrm{e}^{\mathrm{i}(j|\phi)}\, \mathrm{d}\mu(\phi)$ is its Fourier transform (partition function in QFT).

Ideology of Euclidean Wilsonian renormalization

- Take an Euclidean action $S=T+V$, with kinetic + potential term splitting. Say, $T(\varphi)=\frac{1}{2}\int\varphi\,(-\Delta+m^2)\varphi$, and $V(\varphi)=g\int\varphi^4$.

- Then T, i.e. $(-\Delta+m^2)$ has a propagator $K(\cdot,\cdot)$ which is positive definite:
  - $(-\Delta+m^2)_x\,K(x,y)=\delta_y(x)$,
  - for all $j\in\mathcal{D}$ test sources: $(K|j\otimes j)\geq 0$.

- Due to above, the function $Z_T(j):=\mathrm{e}^{-\frac{1}{2}(K|j\otimes j)}$ $(j\in\mathcal{D})$ has "quite nice" properties.

- Bochner-Minlos theorem: because of
  - "quite nice" properties of Z_T,
  - "quite nice" properties of the space $\mathcal{D}$,

  $\exists|$ probability measure γ on $\mathcal{D}'$, whose Fourier transform is Z_T.
  It is the Feynman measure for free theory: $\int\limits_{\phi\in\mathcal{D}'}(\ldots)\,\mathrm{d}\gamma(\phi)=\int\limits_{\phi\in\mathcal{D}'}(\ldots)\,\mathrm{e}^{-T(\phi)}\,\text{“}\mathrm{d}\phi\text{”}$.

- Tempting definition for Feynman measure of interacting theory:

$$\int\limits_{\phi\in\mathcal{D}'}(\ldots)\,\mathrm{e}^{-V(\phi)}\,\mathrm{d}\gamma(\phi)\qquad\left[=\int\limits_{\phi\in\mathcal{D}'}(\ldots)\,\underbrace{\mathrm{e}^{-(T(\phi)+V(\phi))}}_{=\mathrm{e}^{-S(\phi)}}\,\text{“}\mathrm{d}\phi\text{”}\right]$$

- Problem, the interacting Feynman measure $\mu := \mathrm{e}^{-V} \cdot \gamma$ is undefined:

$$\int_{\phi\in\mathcal{D}'} (\ldots) \underbrace{\mathrm{d}\mu(\phi)}_{\text{wannabe Feynman measure}} := \int_{\phi\in\mathcal{D}'} (\ldots) \underbrace{\mathrm{e}^{-V(\phi)}}_{\text{lives on function sense fields}} \underbrace{\mathrm{d}\gamma(\phi)}_{\text{lives on distribution sense fields}}$$

  Because V is spacetime integral of pointwise product of fields, e.g. $V(\varphi) = g\int\varphi^4$. How to bring e^{-V} and γ to common grounds?

- Physicist workaround: Wilsonian regularization.
  Take a continuous linear mapping C: (distributional fields) $\to$ (function sense fields).
  Take the pushforward Gaussian measure $\gamma_C := C_*\gamma$ ← lives on $\mathrm{Ran}(C)$
  Those are functions, so safe to integrate e^{-V} there:

$$\int_{\varphi\in\mathrm{Ran}(C)} (\ldots)\ \mathrm{e}^{-V(\varphi)}\ \mathrm{d}\gamma_C(\varphi) \qquad \left[= \int_{\varphi\in\mathrm{Ran}(C)} (\ldots)\ \mathrm{e}^{-(T_C(\varphi)+V(\varphi))}\ \text{“}\mathrm{d}\varphi\text{”}\right]$$

  ($\varphi\in\mathrm{Ran}(C)$ ← a space of UV regularized fields)

  [Schwartz kernel theorem: C is convolution by a test function, if translationally invariant. I.e., it is a momentum space damping, or coarse-graining of fields.]

- What do we do with the C-dependence? What is the physics / mathematics behind?
- Take a family V_C ($C \in \{$coarse-grainings$\}$) of interaction terms. $\leftrightarrow \mu_C := \mathrm{e}^{-V_C} \cdot \gamma_C$
  We say that it is a Wilsonian renormalization group (RG) flow iff:
  $\exists$ some continuous functional z : $\{$coarse-grainings$\} \to \mathbb{R}$, such that
  $\forall$ coarse-grainings C, C', C'' with $C'' = C'\,C$:
  $$z(C'')_*\,\mu_{C''} \;=\; z(C)_*\,C'_*\,\mu_C$$
  [z is called the running wave function renormalization factor.]
- If $\mathcal{G}_C = (\mathcal{G}_C^{(0)}, \mathcal{G}_C^{(1)}, \mathcal{G}_C^{(2)}, \dots)$ are the moments of μ_C, then
  $\exists$ some continuous functional z : $\{$coarse-grainings$\} \to \mathbb{R}$, such that
  $\forall$ coarse-grainings C, C', C'' with $C'' = C'\,C$:
  $$z(C'')^n\,\mathcal{G}_{C''}^{(n)} \;=\; z(C)^n \otimes^n C'\,\mathcal{G}_C^{(n)} \quad \text{for all } n = 0, 1, 2, \dots$$
  [Valid also in Lorentz signature and on manifolds, for formal moments (correlators).]

[We can always set $z(C) = 1$, by rescaling fields: $\tilde{\mu}_C := z(C)_*\,\mu_C$ or $\tilde{\mathcal{G}}_C^{(n)} := z(C)^n\,\mathcal{G}_C^{(n)}$.]

Part I:

On Wilsonian RG flow of correlators (arbitrary signature, on manifolds)

[*Class.Quant.Grav.* **41** (2024) 125009]

Wilsonian RG flow of correlators, rigorously

Definition:

A continuous linear operator $C:$ (distributional fields) $\to$ (smooth fields)
is coarse-graining iff properly supported and injective on compactly supported distributions.
[Info: on $\mathbb{R}^N$, convolution by test functions are the translationally invariant coarse-grainings.]

A family of smooth correlators $\mathcal{G}_C$ ($C \in$ coarse-grainings) is Wilsonian RG flow iff
$\forall$ coarse-grainings C, C', C'' with $C'' = C'\,C$ one has that
$\mathcal{G}^{(n)}_{C''} = \otimes^n C'\,\mathcal{G}^{(n)}_C$ holds $(n = 0, 1, 2, ...)$. $\longleftarrow$ rigorous RGE in any signature

Space of Wilsonian RG flows is nonempty:
For any distributional correlator G, the family

$$\mathcal{G}^{(n)}_C \quad := \quad \otimes^n C\,G^{(n)} \qquad (*)$$

is a Wilsonian RG flow.

Theorem[A.Lászó, Z.Tarcsay *Class.Quant.Grav.***41**(2024)125009]:
1. On manifolds it is “quite nice” topological vector space, similar to distributions.
2. On flat spacetime for bosonic fields, all Wilsonian RG flows are of the form of $(*)$.

$\downarrow$

UV limit.

Part II:

On Wilsonian RG flows of Feynman measures (Euclidean signature, flat spacetime, bosonic fields)

[**arXiv:2502.16319**]

Wilsonian renormalization in Euclidean signature

We study Euclidean Feynman measures on flat spacetime, for bosonic fields.
[We work on $\mathcal{S}$ and $\mathcal{S}'$, because we can — and also a useful theorem holds there.]

Coarse-grainings: convolution $C_\eta = \eta \star (\cdot)$ by some $\eta \in \mathcal{S}$ Schwartz functions.

One may even restrict η such that:
$0 \leq F(\eta) \leq 1$ and that $F(\eta)$ is unity around zero frequency:

(The proofs go through with that as well.)

Take a family V_C ($C \in \{$coarse-grainings$\}$) of interaction terms $\leftrightarrow$ $\mu_C := \mathrm{e}^{-V_C} \cdot \gamma_C$.
May also allow the coeffs of $(-d^2\Delta + m^2)$ inducing γ to be C-dependent.
Let it be a Wilsonian RG flow:

$\forall$ coarse-grainings C, C', C'' with $C'' = C'\,C$:

$$\mu_{C''} \quad = \quad C'_*\,\mu_C$$

Space of Wilsonian RG flow of Feynman measures is nonempty:

For any probability measure μ on $\mathcal{S}'$, the family

$$\mu_C \quad := \quad C_*\,\mu \qquad (*)$$

is a Wilsonian RG flow.

Theorem[A.Lászó, Z.Tarcsay, J.Ziebell **`arXiv:2502.16319`**]:

1. On flat spacetime for bosonic fields, all Wilsonian RG flows are of the form $(*)$. $\leftarrow$ UV limit
2. Parameters of the reference kinetic Gaussian γ cannot run. $\leftarrow$ not even for eff.field theory!
3. There exists some measurable potential $V : \mathcal{S}' \to \overline{\mathbb{R}}$, such that $\mu = \mathrm{e}^{-V} \cdot \gamma$.
4. For C, with nowhere vanishing Fourier spectrum, one has $\mathrm{e}^{-V_C \circ C} \cdot \gamma = \mathrm{e}^{-V} \cdot \gamma$.
5. If $V_C : C[\mathcal{S}'] \to \overline{\mathbb{R}}$ bounded from below at such C, then V is γ-ess.bounded from below.

What really the Wilsonian RG is about?

Original problem:

- We had $\mathcal{V}:\ \{\text{function sense fields}\}\to\overline{\mathbb{R}}$, say $\mathcal{V}(\varphi)=g\int\varphi^4$.
- We would need to integrate it against γ, but that lives on $\mathcal{S}'$ fields.
- γ known to be supported “sparsely”, i.e. not on function fields, but really on $\mathcal{S}'$.
- So, we really need to extend $\mathcal{V}$ at least γ-a.e. to make sense of $\mu:=\mathrm{e}^{-V}\cdot\gamma$.

Caution by physicists: this may be impossible.

- We are afraid that V on $\mathcal{S}'$ might not exist.
- Instead, let us push γ to smooth fields by C, do there $\mu_C:=\mathrm{e}^{-V_C}\cdot\gamma_C$.
- Then, get rid of C-dependence of μ_C by concept of Wilsonian RG flow.
  Maybe even $\mu_C\to\mu$ could exist as $C\to\delta$ if we are lucky...

Our result: we are back to the start.

- The UV limit Feynman measure μ then indeed exists.
- But we just proved that then there must exist some V on $\mathcal{S}'$ (γ-a.e.) associated to $\mathcal{V}$.
- So, we'd better look for that ominous V.
- For bounded from below $\mathcal{V}$, bounded from below measurable V needed.
  If we find one, $\mu:=\mathrm{e}^{-V}\cdot\gamma$ is then finite measure automatically.
  Only pathology: overlap integral of e^{-V} and γ expected small, maybe zero.
  We only need to make sure that $\int_{\phi\in\mathcal{S}'}\mathrm{e}^{-V(\phi)}\,\mathrm{d}\gamma(\phi)>0$!

A natural extension[A.László, Z.Tarcsay, J.Ziebell **`arXiv:2502.16319`**]:

If $\mathcal{V}$ is bounded from below, there is an optimal extension, the "greedy" extension.

$$V(\cdot) \;:=\; \underset{\{\eta_n\to\delta\}}{(\gamma)\mathrm{inf}}\;\liminf_{\eta_n\to\delta}\;\mathcal{V}(\eta_n\star\;\cdot\;)$$

This is the lower envelope of extensions, i.e. overlap of e^{-V} and γ largest.

We used $(\gamma)\mathrm{inf}$ trick to make V measurable.

Theorem[A.László, Z.Tarcsay, J.Ziebell **`arXiv:2502.16319`**]:

The interacting Feynman measure $\mu := \mathrm{e}^{-V}\cdot\gamma$ by greedy extension is nonzero iff

$$\exists\;\eta_n\to\delta\;:\qquad \int_{\phi\in\mathcal{S}'}\limsup_{n\to\infty}\;\mathrm{e}^{-\mathcal{V}(\eta_n\star\phi)}\;\mathrm{d}\gamma(\phi)\quad>\quad 0.$$

Sufficient condition:

$$\exists\;\eta_n\to\delta\;:\qquad \lim_{n\to\infty}\int_{\phi\in\mathcal{S}'}\mathrm{e}^{-\mathcal{V}(\eta_n\star\phi)}\;\mathrm{d}\gamma(\phi)\quad>\quad 0.$$

Makes μ for all bounded $\mathcal{V}$ meaningful, $\exists$ Schwinger distributions [see *CMP***406**(2025)211].
[Weidling, sine-Gordon yes. φ^4 no. $(\varphi^2-\psi^2)^2$ maybe.]

Summary

- Wilsonian RG flow of correlators can be defined in any signature and on manifolds.
  - Have nice function space properties like distributions.
  - Under mild conditions,
    come from a distributional correlator (UV limit).

- In Euclidean signature, similar for Feynman measures.
  - $\exists$ UV limit Feynman measure.
  - Reference Gaussian cannot run.
  - $\exists$ UV limit interaction potential.
  - A new condition for Wilsonian renormalizability.

Backup slides

Sketch of proofs for part I

1. On manifolds it is "quite nice" topological vector space, similar to distributions.
   [It is Hausdorff, locally convex, complete, nuclear, semi-Montel, Schwartz.]

- Coarse-grainings have a natural ordering of being less UV than an other:
  $C'' \preceq C$ iff $C''=C$ or $\exists\, C'$: $C''=C'\,C$.
- With this, the space of Wilsonian RG flows is seen to be projective limit of copies of $\mathcal{T}(\mathcal{E})$.
- Check known properties of $\mathcal{T}(\mathcal{E})$, some of them are preserved by projective limit.

2. On flat spacetime for bosonic fields, all Wilsonian RG flows are $\mathcal{G}^{(n)}_{C} = \otimes^n C\, G^{(n)}$.

- On flat spacetime, convolution ops by test functions $C_\eta := \eta \star (\cdot)$ exist and commute.
- Due to RGE, commutativity of convolution ops, and polarization formula for n-forms, for bosonic fields $\mathcal{G}^{(n)}_{C_\eta}$ is n-order homogeneous polynomial in η.
  That is, $\exists|$ $\mathcal{G}^{(n)}_{\eta_1,\dots,\eta_n}$ symmetric n-linear map in $\eta_1,\dots,\eta_n$, such that $\mathcal{G}^{(n)}_{C_\eta} = \mathcal{G}^{(n)}_{\eta,\dots,\eta}$.
- Due to RGE, commutativity of convolution ops, and a Banach-Steinhaus thm variant, $\mathcal{G}^{(n)}_{\eta_1^t,\dots,\eta_n^t}\Big|_0$ extends to an n-variate distribution, it will do the job as $(G^{(n)}\,|\,\eta_1\otimes\dots\otimes\eta_n)$.

[A Banach-Steinhaus theorem variant (the key lemma – A.László, Z.Tarcsay):
If a sequence of n-variate distributions pointwise converge on $\otimes^n\mathcal{D}$, then also on full $\mathcal{D}_n$.]

So, it turns out that Wilsonian RG flow of correlators $\leftrightarrow$ distributional correlators.
(under mild conditions)

Executive summary:

- In QFT, the fundamental objects of interest are distributional field correlators.
- Physical ones selected by a “field equation”, the master Dyson-Schwinger equation. Through their smoothed (Wilsonian regularized) instances [*CQG* **39**(2022)185004].

Academic question:

- What about Wilsonian RG flow of measures? (In Euclidean signature QFT.) [**`arXiv:2502.16319`**]

Sketch of proofs for part II

1. On flat spacetime for bosonic fields, all Wilsonian RG flows are of the form $\mu_C = C_* \, \mu$.

- We prove it for Fourier transforms (partition functions), and then use Bochner-Minlos. We use that $\mathcal{S} \star \mathcal{S} = \mathcal{S}$, moreover that for all $\mathcal{J} \subset \mathcal{S}$ compact $\exists \, \eta \in \mathcal{S}$ and $\mathcal{L} \subset \mathcal{S}$ compact such that $\mathcal{J} = \eta \star \mathcal{L}$.

2. Parameters of the reference kinetic Gaussian γ cannot run.

- Pushforward preserves abs.continuity, plus a rigidity property of Gaussian measures.

3. There exists some measurable potential $V : \mathcal{S}' \to \overline{\mathbb{R}}$, such that $\mu = \mathrm{e}^{-V} \cdot \gamma$.

- We apply Radon-Nikodym theorem, the fact that $\mathcal{S}'$ is so-called Souslin space, and that for $\eta \in \mathcal{S}$ with $F(\eta) > 0$ the coarse-graining $C_\eta := \eta \star (\cdot)$ is injective.

4. For C, with nowhere vanishing Fourier spectrum, one has $\mathrm{e}^{-V_C \circ C} \cdot \gamma = \mathrm{e}^{-V} \cdot \gamma$.

- Fundamental formula of integration variable substitution vs pusforward, Souslin-ness of $\mathcal{S}'$, injectivity of coarse-graining $C_\eta := \eta \star (\cdot)$ with $\eta \in \mathcal{S}$, $F(\eta) > 0$.

Relation to usual RG theory:

Fix some $\eta \in \mathcal{S}$ such that $\int \eta = 1$ and $F(\eta) > 0$.
Introduce scaled η, that is $\eta_\Lambda(x) := \Lambda^N \eta(\Lambda\, x)$ (for all $x \in \mathbb{R}^N$ and scaling $1 \leq \Lambda < \infty$).
One has $\eta_\Lambda \xrightarrow{\mathcal{S}'} \delta$ as $\Lambda \longrightarrow \infty$.

By our theorem, for all Λ, one has $\mathrm{e}^{-V_{C_{\eta_\Lambda}} \circ C_{\eta_\Lambda}} \cdot \gamma = \mathrm{e}^{-V} \cdot \gamma$.

$\Downarrow$

Informally: ODE for $V_{C_{\eta_\Lambda}}$, namely $\frac{\mathrm{d}}{\mathrm{d}\Lambda} \left(\mathrm{e}^{-V_{C_{\eta_\Lambda}} \circ C_{\eta_\Lambda}} \cdot \gamma \right) = 0$ for $1 \leq \Lambda < \infty$.

QFT people try to solve such flow equation, given initial data $V_{C_\Lambda}\big|_{\Lambda=1}$.

But why bother? By our theorem, all RG flows of such kind has some V at the UV end.
Look directly for V?

Case of strictly bandlimited momentum cutoff

Some QFT literatures postulate that Fourier profile of regulators are strictly bandlimited:

Sharp bandlimited momentum cutoff (in a tricky way) can also be defined, γ -a.e.:

What stays true from our theorems?

1. On flat spacetime for bosonic fields, if μ_C -s have second moment, $\exists$ UV measure μ.
2. Parameters of kinetic $(-d^2\Delta + m^2)$, to which γ is associated, cannot run.
   (Not even for terminating flows! That is, also for effective field theories.)

   Don't know: if existence of UV limit potential stays true.

Info: bandlimiting not meaningful on manifolds, "not natural".

On coarse-grainings

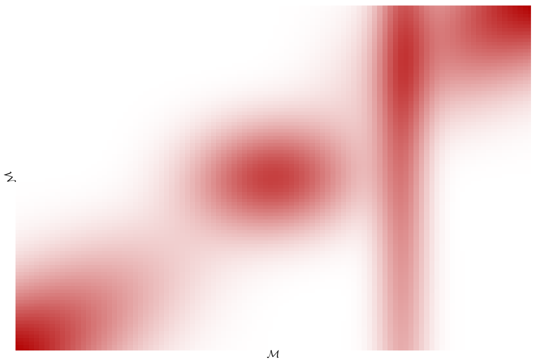
Original problematics: need to make functions of distributions, so that $\int_{\mathcal{M}} \varphi^4$ is meaningful.

Let $C :\ \mathcal{E}' \to \mathcal{E}$ continuous linear mapping.

Schwartz kernel theorem:

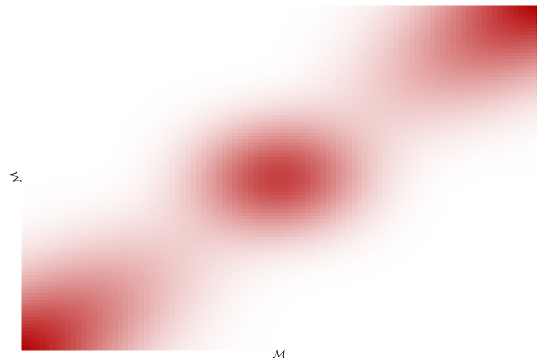
$\exists\ \kappa(x,y)$ smooth sect on $\mathcal{M}\times\mathcal{M}$:

$\forall\, T\in\mathcal{E}'$: $(C\,T)\big|_y = (T\,|\,\kappa(\cdot,y))$ holds.

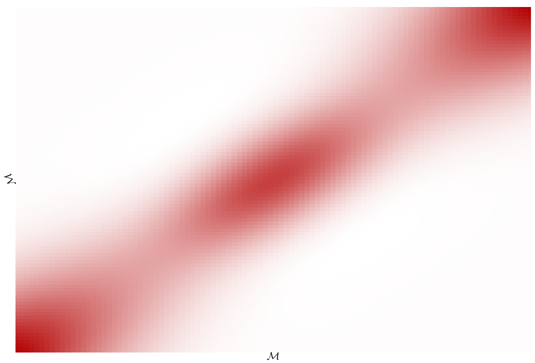


Let C_κ preserve compact support, also C_κ^t !

Then, extendable as $C_\kappa : \mathcal{D}' \to \mathcal{E}$.

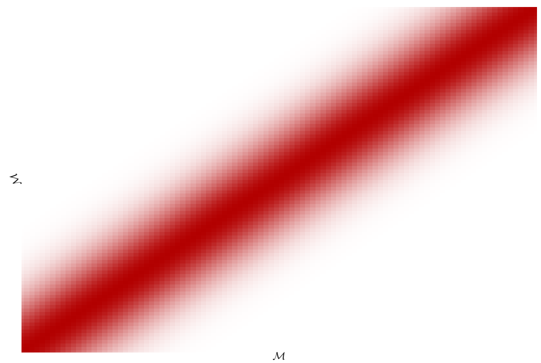


Let C_κ injective on $\mathcal{E}'$, also C_κ^t !



If $\mathcal{M}\equiv\mathbb{R}^N$, let C_κ be translation invariant.

Then, $\kappa(x,y)=\eta(x-y)$ for some $\eta\in\mathcal{D}\setminus\{0\}$.



That is, $C_\kappa = C_\eta = \eta\star(\cdot)$ for some $\eta\in\mathcal{D}\setminus\{0\}$.

On flat spacetime we can also play this on Schwartz distributions:
let $C:\ \mathcal{S}'\to\mathcal{E}$ cont.lin. mapping, and $C^t:\ \mathcal{E}'\to\mathcal{S}\subset\mathcal{E}$ to be $\mathcal{S}'$-cont, and transl.inv.
Then: $C_\kappa=C_\eta=\eta\star(\cdot)$ for some $\eta\in\mathcal{S}\setminus\{0\}$.

Customary to make further restrictions on $F(\eta)$.

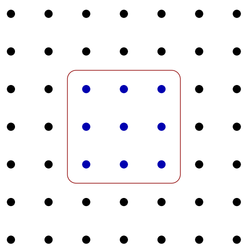
E.g. flat unity top + Schwartz tail.

These admit ones e.g. with Gaussian tail.

Sometimes, smooth band limited tail is required.

Extreme case: not trivial, but sharp cutoff is γ-a.e. meaningful.

On manifolds, these are not meaningful.
Not natural even on flat spacetime:



Lattice averaging corresponds to C_η with $\eta\in\mathcal{D}\setminus\{0\}$.
But Paley-Wiener-Schwartz theorem:
then $F(\eta)$ is not bandlimited.

Some complications on topological vector spaces

Careful with tensor algebra! Schwartz kernel theorems:

$$\hat{\otimes}_\pi^n \mathcal{E} \equiv \mathcal{E}_n \equiv (\hat{\otimes}_\pi^n \mathcal{E}')' \equiv \mathcal{L}in(\mathcal{E}', \hat{\otimes}_\pi^{n-1} \mathcal{E})$$

$$(\hat{\otimes}_\pi^n \mathcal{E})' \equiv \mathcal{E}'_n \equiv \hat{\otimes}_\pi^n \mathcal{E}' \equiv \mathcal{L}in(\mathcal{E}, \hat{\otimes}_\pi^{n-1} \mathcal{E}')$$

$$\hat{\otimes}_\pi^n \mathcal{D} \underset{\text{cont.bij.}}{\leftarrow} \mathcal{D}_n \equiv (\hat{\otimes}_\pi^n \mathcal{D}')'$$

$$(\hat{\otimes}_\pi^n \mathcal{D})' \rightarrow \mathcal{D}'_n \equiv \hat{\otimes}_\pi^n \mathcal{D}' \equiv \mathcal{L}in(\mathcal{D}, \hat{\otimes}_\pi^{n-1} \mathcal{D}')$$

$\mathcal{E} \times \mathcal{E} \rightarrow F$ separately continuous maps are jointly continuous.
$\mathcal{E}' \times \mathcal{E}' \rightarrow F$ separately continuous bilinear maps are jointly continuous.
For mixed, no guarantee.
For $\mathcal{D}$ or $\mathcal{D}'$ spaces, joint continuity from separate continuity of bilinear forms not automatic.
For mixed, even less guarantee.

But as convergence vector spaces, everything is nice with mixed $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{D}$, $\mathcal{D}'$ multilinears (separate sequential continuity $\Leftrightarrow$ joint sequential continuity).