On the equation of motion of quantum field theory

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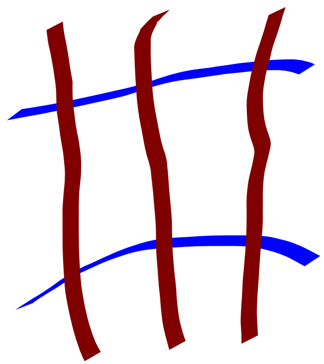
Structure of model building in fundamental physics

Relativistic or non-relativistic point mechanics:

- Take Newton equation over a fixed spacetime and fixed potentials.
- Space of solutions turns out to be a symplectic manifold.
- One can play classical probability theory on the solution space.



Relativistic or non-relativistic quantum mechanics:

- Take Dirac etc. equation over a fixed spacetime and fixed potentials.
- Space of finite charge weak solutions turns out to be a Hilbert space.
- One can play quantum probability theory on the solution space.

Most important ingredient: a one-liner, the equation of motion or field equation.
$\longrightarrow$ Then, one is working on the solution space.

Can one find a one-liner equation to summarize Quantum Field Theory?

Model building attempts in QFT

Common QFT formalisms in physics:

- Often non-manifestly covariant formalism.
  (Hamiltonian, reminescents of non-relativistic QM as seen by an inertial observer.)
- In momentum space.
- Splitting Lagrangian to free + interacting terms.
- Often perturbative handling.
- Need for regularization and renormalization. (What this is precisely?)
- Not easy to see what is legitimate and what is not.
- In some cases the “right” thing is done, even without the adequate formalism.

Common formalisms in mathematical QFT:

- Loop quantum gravity. (Spacetime is emergent, far from finalized.)
- Algebraic QFT: easy to understand math/physics concept, no known 3+1d example.
- Perturbative AQFT formalized over fixed spacetime, known examples.
  (Still cannot put down a one-liner.)
- Feynman integral in Wick rotated signature. But still free + interaction splitting.

Followed guidelines

Do not use (unless emphasized):

- Structures specific to an affine spacetime manifold.
- Known fixed spacetime metric / causal structure.
- Known splitting of Lagrangian to free + interaction term.

Consequences:

- Cannot go to momentum space, have to stay in spacetime description.
- Cannot refer to any affine property of Minkowski spacetime, e.g. asymptotics. (No Schwartz functions.)
- Cannot use Wick rotation to Euclidean signature metric.
- Even if Wick rotated, no free + interaction splitting, so no Gaussian Feynman measure.
- Can only use generic, differential geometrically natural objects.

Outline

Will attempt to set up eom for the key ingredient for the quantum probability space of QFT.

I. On Wilsonian regularized Feynman functional integral formulation:

- Can be substituted by regularized master Dyson-Schwinger equation for correlators.
- For conformally invariant or flat spacetime Lagrangians, showed an existence condition for regularized MDS solutions, provides convergent iterative solver method.

[*Class.Quant.Grav.***39**(2022)185004]

II. On Wilsonian renormalization group flows of correlators:

- They form a topological vector space which is
Hausdorff, locally convex, complete, nuclear, semi-Montel, Schwartz.
- On flat spacetime for bosonic fields: in bijection with distributional correlators.

[**`arXiv:2303.03740`** *with Zsigmond Tarcsay*]

Part I:

On Wilsonian regularized Feynman functional integral formulation

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Test field variations: $\delta\psi_T \in \mathcal{D}$, compactly supported ones from $\mathcal{E}$ with $\mathcal{D}$ test funct. top.

Informal Feynman functional integral in Lorentz signature

Fix a reference field $\psi_0 \in \boldsymbol{\mathcal{E}}$ for bringing the problem from $\boldsymbol{\mathcal{E}}$ to $\mathcal{E}$, and take $J_1, ..., J_n \in \mathcal{E}'$.
Then, $\psi \mapsto (J_1|\psi{-}\psi_0) \cdot ... \cdot (J_n|\psi{-}\psi_0)$ defines a $\boldsymbol{\mathcal{E}} \to \mathbb{R}$ polynomial observable.

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Feynman type quantum vacuum expectation value of this is postulated as:

$$\int\limits_{\psi\in\boldsymbol{\mathcal{E}}} (J_1|\psi-\psi_0) \cdot \ldots \cdot (J_n|\psi-\psi_0) \;\; \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(\psi)}\,\mathrm{d}\lambda(\psi) \Bigg/ \int\limits_{\psi\in\boldsymbol{\mathcal{E}}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(\psi)}\,\mathrm{d}\lambda(\psi)$$

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Partition function often invoked to book-keep these (formal Fourier transform of $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\,\lambda$):

$$Z_{\psi_0}: \quad \mathcal{E}' \longrightarrow \mathbb{C}, \quad J \longmapsto Z_{\psi_0}(J) := \int_{\psi\in\boldsymbol{\mathcal{E}}} \mathrm{e}^{\mathrm{i}\,(J|\psi-\psi_0)} \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(\psi)} \, \mathrm{d}\lambda(\psi),$$

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and from this one can define

$$G^{(n)}_{\psi_0} \quad := \quad \left((-\mathrm{i})^n \, \frac{1}{Z_{\psi_0}(J)} \, \partial^{(n)}_J Z_{\psi_0}(J) \right) \Bigg|_{J=0}$$

n-field correlator, and their collection $G_{\psi_0} := (G^{(0)}_{\psi_0}, G^{(1)}_{\psi_0}, \ldots, G^{(n)}_{\psi_0}, \ldots) \in \bigoplus\limits_{n\in\mathbb{N}_0} \overset{n}{\otimes}\,\mathcal{E}$.

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Above quantum expectation value expressable via distribution pairing: $\big(J_1\otimes\ldots\otimes J_n \,\big|\, G^{(n)}_{\psi_0}\big)$.

Well known problems:

- No “Lebesgue” measure λ in infinite dimensions.
- Neither $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\,\lambda$ is meaningful. (Can be repaired to some extent in Euclidean signature.)
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- as if λ existed as *translation invariant* (Lebesgue) measure,
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Textbook “theorem”: because of above rules, one has
$Z:\,\mathcal{E}'\to\mathbb{C}$ is Fourier transform of $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\,\lambda$ “$\Longleftrightarrow$” it satisfies master-Dyson-Schwinger eq

$$\Big(\,\mathbf{E}\big((-\mathrm{i})\partial_J+\psi_0\big)\;+\;\hbar\,J\,\Big)Z(J)\quad=\quad 0\quad(\forall J\in\mathcal{E}')$$

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Does this informal PDE have a meaning? [Yes, on the correlators $G = \big(G^{(0)}, G^{(1)}, \ldots\big)$.]

Rigorous definition of Euler-Lagrange functional

- Let a Lagrange form be given, which is

$$\mathrm{L}: \; V(\mathcal{M}) \;\oplus\; T^*(\mathcal{M})\otimes V(\mathcal{M}) \;\oplus\; T^*(\mathcal{M})\wedge T^*(\mathcal{M})\otimes V(\mathcal{M})\otimes V^*(\mathcal{M}) \;\longrightarrow\; \overset{\dim(\mathcal{M})}{\wedge} T^*(\mathcal{M})$$

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- Action functional:

$$S:\quad \underbrace{\Gamma\big(V(\mathcal{M}) \times_{\mathcal{M}} \mathrm{CovDeriv}(V(\mathcal{M}))\big)}_{=:\;\boldsymbol{\mathcal{E}}} \longrightarrow \mathrm{Meas}(\mathcal{M},\mathbb{R}), \quad \underbrace{(v,\nabla)}_{=:\;\psi} \longmapsto \Big(\mathcal{K}\mapsto S_{\mathcal{K}}(v,\nabla)\Big)$$

where $S_{\mathcal{K}}(v,\nabla) := \int_{\mathcal{K}} \mathrm{L}(v,\nabla v, F(\nabla))$ for all $\mathcal{K}\subset\mathcal{M}$ compact.

Action functional $S : \boldsymbol{\mathcal{E}} \to \mathrm{Meas}(\mathcal{M}, \mathbb{R})$ Fréchet differentiable, its Fréchet derivative

$$DS : \quad \boldsymbol{\mathcal{E}} \times \mathcal{E} \longrightarrow \mathrm{Meas}(\mathcal{M}, \mathbb{R}), \quad (\psi, \delta\psi) \longmapsto \Big(\mathcal{K} \mapsto \big(DS_{\mathcal{K}}(\psi) \,\big|\, \delta\psi\big)\Big)$$

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Observables are the $O : \boldsymbol{\mathcal{E}} \to \mathbb{R}$ continuous maps.

Rigorous definition of master Dyson-Schwinger equation

- Want to rephrase informal MDS operator on Z to n-field correlators $G = \big(G^{(0)}, G^{(1)}, ...\big)$.
  These sit in the tensor algebra $\mathcal{T}(\mathcal{E}) := \bigoplus_{n\in\mathbb{N}_0} \hat{\otimes}_{\pi}^{n}\,\mathcal{E}$ of field variations.
  More precisely, they sit in a graded-symmetrized subspace, e.g. $\bigvee(\mathcal{E})$ or $\bigwedge(\mathcal{E})$ of $\mathcal{T}(\mathcal{E})$.
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- One has $\big(\mathcal{T}(\mathcal{E})\big)'\equiv\mathcal{T}_a(\mathcal{E}')$ and $\big(\mathcal{T}(\mathcal{E})\big)''\equiv\mathcal{T}(\mathcal{E})$ etc, “nice” properties.
  Moreover, tensor algebra of field variations is topological unital bialgebra.

Rigorous definition of master Dyson-Schwinger equation

- Want to rephrase informal MDS operator on Z to n-field correlators $G=\left(G^{(0)},G^{(1)},\ldots\right)$.
  These sit in the tensor algebra $\mathcal{T}(\mathcal{E}):=\bigoplus_{n\in\mathbb{N}_0}\hat{\otimes}_\pi^n\,\mathcal{E}$ of field variations.
  More precisely, they sit in a graded-symmetrized subspace, e.g. $\bigvee(\mathcal{E})$ or $\bigwedge(\mathcal{E})$ of $\mathcal{T}(\mathcal{E})$.
  Naturally topologized: with Tychonoff topology, similar to $\mathcal{E}$, i.e. nuclear Fréchet.

- Algebraic tensor algebra $\mathcal{T}_a(\mathcal{E}'):=\oplus_{n\in\mathbb{N}_0}\hat{\otimes}_\pi^n\,\mathcal{E}'$ of sources.
  Naturally topologized: loc.conv. direct sum topology, similar to $\mathcal{E}'$, i.e. dual nuclear Fréchet.

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  Moreover, tensor algebra of field variations is topological unital bialgebra.
  Unity $\mathbb{1}:=(1,0,0,0,\ldots)$.
  Left-multiplication L_x by a fix element x meaningful and continuous linear.
  Left-insertion ι_p (tracing out) by $p\in\left(\mathcal{T}(\mathcal{E})\right)'\equiv\mathcal{T}_a(\mathcal{E}')$ also meaningful, continuous linear.
  Usual graded-commutation: $\left(\iota_p\,L_{\delta\psi}\pm L_{\delta\psi}\,\iota_p\right)G\;=\;(p|\delta\psi)\,G\quad(\forall p\in\mathcal{E}',\ \delta\psi\in\mathcal{E},\ G)$.

Take a classical observable $O : \boldsymbol{\mathcal{E}} \to \mathbb{R}$, $\psi \mapsto O(\psi)$, let $O_{\psi_0} := O \circ (\mathrm{I}_{\mathcal{E}} + \psi_0)$.

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We say that O is multipolynomial iff for some $\psi_0\in\boldsymbol{\mathcal{E}}$ there exists $\mathbf{O}_{\psi_0}\in\mathcal{T}_a(\mathcal{E}')$, such that

$$\forall\,\psi\in\boldsymbol{\mathcal{E}}:\quad \underbrace{O_{\psi_0}(\psi{-}\psi_0)}_{=O(\psi)}\;=\;\Big(\mathbf{O}_{\psi_0}\,\Big|\,\big(1,\,\overset{1}{\otimes}(\psi{-}\psi_0),\,\overset{2}{\otimes}(\psi{-}\psi_0),\,\ldots\big)\Big).$$

Similarly $E : \boldsymbol{\mathcal{E}} \to \mathcal{D}'$, $\psi \mapsto E(\psi)$, let $E_{\psi_0} := E \circ (\mathrm{I}_{\mathcal{E}} + \psi_0)$ the same re-expressed on $\mathcal{E}$.

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For fixed $\delta\psi_T\in\mathcal{D}$ one has $(\mathbf{E}_{\psi_0}\,|\,\delta\psi_T)\in\mathcal{T}_a(\mathcal{E}')$, i.e. one can left-insert with it:

$\iota_{(\mathbf{E}_{\psi_0}\,|\,\delta\psi_T)}$ meaningfully acts on $\mathcal{T}(\mathcal{E})$.

The master Dyson-Schwinger (MDS) equation is:

$$\text{we search for } (\psi_0, G_{\psi_0}) \text{ such that: } \qquad \underbrace{G^{(0)}_{\psi_0}}_{=:\, b\, G_{\psi_0}} \quad = \quad 1,$$

$$\forall\, \delta\psi_T \in \mathcal{D} : \qquad \underbrace{\Big(\ell_{(\mathbf{E}_{\psi_0} \,|\, \delta\psi_T)} \; - \; \mathrm{i}\, \hbar\, L_{\delta\psi_T} \Big)}_{=:\, \mathbf{M}_{\psi_0, \delta\psi_T}} \, G_{\psi_0} \quad = \quad 0.$$

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[Feynman type quantum vacuum expectation value of O is then $\big(\mathbf{O}_{\psi_0} \,\big|\, G_{\psi_0}\big)$.]

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$$E: \quad \boldsymbol{\mathcal{E}} \times \mathcal{D} \longrightarrow \mathbb{R}, \quad (\psi,\, \delta\psi_{T}) \longmapsto \int\limits_{y\in\mathcal{M}} \delta\psi_{T}(y)\,\Box_y\psi(y)\,\mathrm{v}(y) \;+\; \int\limits_{y\in\mathcal{M}} \delta\psi_{T}(y)\,\psi^3(y)\,\mathrm{v}(y).$$

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$$\Downarrow$$

MDS operator at $\psi_0 = 0$ reads

$$\big(\mathbf{M}_{\psi_0,\delta\psi_T}\, G\big)^{(n)}(x_1, \ldots, x_n) =$$

$$\int\limits_{y\in\mathcal{M}} \delta\psi_T(y)\, \Box_y G^{(n+1)}(y, x_1, \ldots, x_n)\, \mathrm{v}(y) \;+\; \int\limits_{y\in\mathcal{M}} \delta\psi_T(y)\, G^{(n+3)}(y, y, y, x_1, \ldots, x_n)\, \mathrm{v}(y)$$

$$-\mathrm{i}\,\hbar\; \underbrace{n\, \tfrac{1}{n!} \textstyle\sum_{\pi\in\Pi_n} \delta\psi_T(x_{\pi(1)})\, G^{(n-1)}(x_{\pi(2)}, \ldots, x_{\pi(n)})}_{=\,(L_{\delta\psi_T} G)^{(n)}(x_1,\ldots,x_n)}$$

Pretty much well-defined, and clear recipe, if field correlators were *functions*.

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Workaround in QFT: Wilsonian regularization using coarse-graining (UV damping).

Wilsonian regularized master Dyson-Schwinger equation

- When $\mathcal{E}$ (resp $\mathcal{D}$) are smooth sections of some vector bundle, denote by $\mathcal{E}^{\times}$ (resp $\mathcal{D}^{\times}$) the smooth sections of its densitized dual vector bundle. Then, distributional sections are $\mathcal{D}^{\times\prime}$ (resp $\mathcal{E}^{\times\prime}$).

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Coarse-graining ops are natural generalization of convolution by test functions to manifolds.

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Brings back problem from distributions to smooth functions, but depends on regulator κ.

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But what we do with κ dependence? (Rigorous Wilsonian renormalization?)

Part II:

On Wilsonian RG flows of correlators

Informal Wilsonian RG flows of Feynman measures

Fix a reference field $\psi_0 \in \boldsymbol{\mathcal{E}}$ to bring the problem from $\boldsymbol{\mathcal{E}}$ to $\mathcal{E}$.

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Assume that one has an action $S_{\psi_0, C_\kappa} : \underbrace{C_\kappa[\mathcal{D}^{\times\,\prime}]}_{\subset\mathcal{E}} \to \mathbb{R}$ for a coarse-graining C_κ.

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A family of actions S_{ψ_0,C_κ} $(C_\kappa \in \text{coarse-grainings})$ is Wilsonian RG flow iff:
$\forall$ coarse-grainings C_κ, C_μ, C_ν with $C_\nu = C_\mu C_\kappa$ one has that
$\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_{\psi_0,C_\nu}}\,\lambda_{C_\nu}$ is the pushforward of $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_{\psi_0,C_\kappa}}\,\lambda_{C_\kappa}$ by C_μ. $\longleftarrow$ RGE

Rigorous definition will be this, but expressed on the formal moments (n-field correlators).

Rigorous Wilsonian RG flows

Definition:

A family of smooth correlators $\mathcal{G}_{C_\kappa}$ $(C_\kappa \in$ coarse-grainings$)$ is Wilsonian RG flow iff

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$\mathcal{G}^{(n)}_{C_\nu} \; = \; \otimes^n C_\mu \, \mathcal{G}^{(n)}_{C_\kappa}$ holds $(n = 0, 1, 2, \ldots)$. $\longleftarrow$ rigorous RGE

Rigorous Wilsonian RG flows

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A family of smooth correlators $\mathcal{G}_{C_\kappa}$ ($C_\kappa \in$ coarse-grainings) is Wilsonian RG flow iff

$\forall$ coarse-grainings C_κ, C_μ, C_ν with $C_\nu = C_\mu C_\kappa$ one has that

$\mathcal{G}^{(n)}_{C_\nu} = \otimes^n C_\mu \, \mathcal{G}^{(n)}_{C_\kappa}$ holds $(n = 0, 1, 2, ...)$. $\longleftarrow$ rigorous RGE

Space of Wilsonian RG flows is nonempty:

For any distributional correlator G, the family

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Space of Wilsonian RG flows naturally topologized by coarse-graining-wise Tychonoff.

Theorem[A.L., Z.Tarcsay]:
1. On manifolds it is Hausdorff, locally convex, complete, nuclear, semi-Montel, Schwartz.
2. On flat spacetime for bosonic fields, all Wilsonian RG flows are of the form of $(*)$.

Sketch of proof for 1.:

- Coarse-grainings have a natural partial ordering of being less UV than an other: $C_\nu \preceq C_\kappa$ iff $C_\nu = C_\kappa$ or $\exists\, C_\mu :\ C_\nu = C_\mu C_\kappa$.
- With this, the space of Wilsonian RG flows is seen to be projective limit of copies of $\mathcal{T}(\mathcal{E})$.
- Check known properties of $\mathcal{T}(\mathcal{E}),$ some of them are preserved by projective limit.

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Sketch of proof for 2.:

- On flat spacetime, convolution ops by test functions $C_f := f \star (\cdot)$ exist and commute.
- Due to RGE, commutativity of convolution ops, and polarization formula for n-forms, for bosonic fields $\mathcal{G}^{(n)}_{C_f}$ is n-order homogeneous polynomial in f.

  That is, one has corresponding $\mathcal{G}^{(n)}_{f_1,\dots,f_n}$ symmetric n-linear map in $f_1, ..., f_n$.
- Due to RGE, commutativity of convolution ops, and an improved Banach-Steinhaus thm, $\mathcal{G}^{(n)}_{f_1^t,\dots,f_n^t}\Big|_0$ extends to an n-variate distribution, which will do the job.

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An improved Banach-Steinhaus theorem (the key lemma – A.L., Z.Tarcsay):
If a sequence of n-variate distributions pointwise converge on $\otimes^n\mathcal{D}$, it does also on full $\mathcal{D}_n$.
Therefore, by ordinary Banach-Steinhaus thm, the limit is an n-variate distribution.

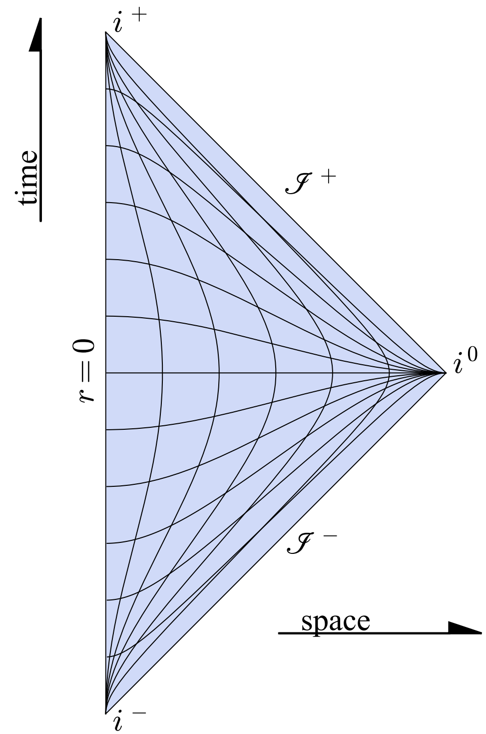
Summary

- Feynman integral has no rigorous definition in Lorentz signature.
- Can be substituted by master Dyson-Schwinger (MDS) equation.
- Function spaces and operators for MDS equation are well defined (in suitable variables).
- Wilsonian regularized version of MDS equation is well defined (in suitable variables).
- Does not need a pre-arranged fixed causal structure.
- Existence condition proved for Wilsonian regularized MDS solutions. Provides a convergent iterative approximation algorithm.
- Space of Wilsonian RG flows of correlators:
  - Hausdorff, locally convex, complete, nuclear, semi-Montel, Schwartz.
  - are in bijection with distributions, on flat spacetime for bosonic fields.

Backup slides

Existence condition for regularized MDS solutions

If Euler-Lagrange functional $E : \boldsymbol{\mathcal{E}} \to \mathcal{D}'$ conformally invariant: re-expressable on Penrose conformal compactification.



That is always a compact manifold, with cone condition boundary.

$E : \boldsymbol{\mathcal{E}} \to \mathcal{D}'$ reformulable over this base manifold.

In such situation, $\mathcal{E} = \mathcal{D}$ and have nice properties:
countably Hilbertian nuclear Fréchet (CHNF) space.

$$F_0 \ \supset \ F_1 \ \supset \ ... \ \supset \ F_m \ \supset \ ... \ \supset \ \mathcal{E}$$

(Intersection of shrinking Hilbert spaces F_m with Hilbert-Schmidt embedding.)

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Theorem [Dubin,Hennings:*P.RIMS***25**(1989)971]:
without penalty, one can equip $\mathcal{T}(\mathcal{E})$ with a better topology, inheriting CHNF topology.

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Regularized MDS operator is then a Hilbert-Schmidt linear map

$$\mathbf{M}_{\psi_0,\kappa} : \quad H_m \otimes F_m \longrightarrow H_0, \quad \mathcal{G} \otimes \delta\psi_T \longmapsto \mathbf{M}_{\psi_0,\kappa,\delta\psi_T}\mathcal{G}$$

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Theorem: one can legitimately trace out $\delta\psi_T$ variable to form

$$\hat{\mathbf{M}}^2_{\psi_0,\kappa} : \quad H_m \longrightarrow H_m, \quad \mathcal{G} \longmapsto \sum_{i\in\mathbb{N}_0} \mathbf{M}^\dagger_{\psi_0,\kappa,\delta\psi_T\,i}\mathbf{M}_{\psi_0,\kappa,\delta\psi_T\,i}\mathcal{G}$$

By construction: $\mathcal{G}$ is κ-regularized MDS solution $\iff$ $b\,\mathcal{G} = 1$ and $\hat{\mathbf{M}}^2_{\psi_0,\kappa}\mathcal{G} = 0$.

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Theorem [A.L.]:

(i) the iteration

$$\mathcal{G}_0 := \mathbb{1} \text{ and } \mathcal{G}_{l+1} := \mathcal{G}_l - \tfrac{1}{T}\,\hat{\mathbf{M}}^2_{\psi_0,\kappa}\mathcal{G}_l \qquad (l = 0, 1, 2, ...)$$

is always convergent if $T >$ trace norm of $\hat{\mathbf{M}}^2_{\psi_0,\kappa}$.

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$$\lim_{l\to\infty} b\,\mathcal{G}_l \;\neq\; 0.$$

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$$\lim_{l\to\infty} \mathcal{G}_l$$

is an MDS solution, up to normalization factor.

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Use for lattice-like numerical method in Lorentz signature?
(Treatment can be adapted to flat spacetime also, because Schwartz functions are CHNF.)

Structure of model building in fundamental physics

Relativistic or non-relativistic point mechanics:

- Take Newton equation over a fixed spacetime and fixed potentials.
- Solution space to the equation turns out to be a symplectic manifold.
- One can play classical probability theory on the solution space:
  - Elements of solution space X are elementary events.
  - Collection of Borel sets Σ of X are composite events.
  - A state is a probability measure W on Σ, i.e. (X, Σ, W) is classical probability space.

Relativistic or non-relativistic quantum mechanics:

- Take Dirac etc. equation over a fixed spacetime and fixed potentials.
- Finite charge weak solution space to the equation turns out to be a Hilbert space.
- One can play quantum probability theory on the solution space:
  - One dimensional subspaces of the solution space $\mathcal{H}$ are elementary events, X.
  - Collection of all closed subspaces Σ of $\mathcal{H}$ are composite events.
  - A state is a probability measure W on Σ, i.e. (X, Σ, W) is quantum probability space.

Fréchet derivative in top.vector spaces

Let F and G real top.affine space, Hausdorff.
Subordinate vector spaces: $\mathbb{F}$ and $\mathbb{G}$.

A map $S:\ F \to G$ is Fréchet-Hadamard differentiable at $\psi \in F$ iff:
there exists $DS(\psi):\ \mathbb{F} \to \mathbb{G}$ continuous linear, such that for all sequence $n \mapsto h_n$ in $\mathbb{F}$, and nonzero sequence $n \mapsto t_n$ in $\mathbb{R}$ which converges to zero,

$$(\mathbb{G})\lim_{n\to\infty}\left(\frac{S(\psi + t_n\, h_n) - S(\psi)}{t_n} - DS(\psi)\, h_n\right) \quad = \quad 0$$

holds.

Fréchet derivative of action functional

Fréchet derivative of $S:\, \boldsymbol{\mathcal{E}} \longrightarrow \mathrm{Meas}(\mathcal{M}, \mathbb{R})$ is

$$DS:\, \boldsymbol{\mathcal{E}} \times \mathcal{E} \longrightarrow \mathrm{Meas}(\mathcal{M}, \mathbb{R}),\; (\psi, \delta\psi) \longmapsto \Big(\mathcal{K} \mapsto \big(DS_{\mathcal{K}}(\psi)\big|\delta\psi\big)\Big)$$

For $\underbrace{(v, \nabla)}_{=:\psi} \in \boldsymbol{\mathcal{E}}$ given,

$$\underbrace{(\delta v, \delta C)}_{=:\delta\psi} \mapsto \big(DS_{\mathcal{K}}(v, \nabla) \,\big|\, (\delta v, \delta C)\big) =$$

$$\int_{\mathcal{K}} \Big(D_1 \mathrm{L}(v, \nabla v, P(\nabla))\, \delta v + D_2^a \mathrm{L}(v, \nabla v, P(\nabla))\, (\nabla_a \delta v + \delta C_a v) + 2\, D_3^{[ab]} \mathrm{L}(v, \nabla v, P(\nabla))\, \tilde{\nabla}_{[a} \delta C_{b]}\Big)$$

$$= \int_{\mathcal{K}} \Big(D_1 \mathrm{L}(v, \nabla v, P(\nabla))_{[c_1 \ldots c_m]}\, \delta v - \big(\tilde{\nabla}_a D_2^a \mathrm{L}(v, \nabla v, P(\nabla))_{[c_1 \ldots c_m]}\big)\, \delta v\Big) +$$

$$\Big(D_2^a \mathrm{L}(v, \nabla v, P(\nabla))_{[c_1 \ldots c_m]}\, \delta C_a v - 2\, \big(\tilde{\nabla}_a D_3^{[ab]} \mathrm{L}(v, \nabla v, P(\nabla))_{[c_1 \ldots c_m]}\big)\, \delta C_b\Big)$$

$$+\; m \int_{\partial\mathcal{K}} \Big(D_2^a \mathrm{L}(v, \nabla v, P(\nabla))_{[a c_1 \ldots c_{m-1}]}\, \delta v + 2\, D_3^{[ab]} \mathrm{L}(v, \nabla v, P(\nabla))_{[a c_1 \ldots c_{m-1}]}\, \delta C_b\Big)$$

$$(m := \dim(\mathcal{M}))$$

[usual Euler-Lagrange bulk integral + boundary integral]

Distributions on manifolds

$W(\mathcal{M})$ vector bundle, $W^{\times}(\mathcal{M}) := W^{*}(\mathcal{M}) \otimes \overset{\dim(\mathcal{M})}{\wedge} T^{*}(\mathcal{M})$ its densitized dual.
$W^{\times\times}(\mathcal{M}) \equiv W(\mathcal{M})$.

Correspondingly: $\mathcal{E}^{\times}$ and $\mathcal{D}^{\times}$ are densitized duals of $\mathcal{E}$ and $\mathcal{D}$.

$\mathcal{E} \times \mathcal{D}^{\times} \to \mathbb{R}$, $(\delta\psi, p_{T}) \mapsto \int\limits_{\mathcal{M}} \delta\psi\, p_{T}$ and $\mathcal{D} \times \mathcal{E}^{\times} \to \mathbb{R}$, $(\delta\psi_{T}, p) \mapsto \int\limits_{\mathcal{M}} \delta\psi_{T}\, p$ jointly sequentially continuous.

Therefore, continuous dense linear injections $\mathcal{E} \to \mathcal{E}^{\times\prime}$ and $\mathcal{D} \to \mathcal{D}^{\times\prime}$.
(hance the name, distributional sections)

Let $A:\ \mathcal{E} \to \mathcal{E}$ continuous linear.
It has formal transpose iff there exists $A^{t}:\ \mathcal{D}^{\times} \to \mathcal{D}^{\times}$ continuous linear, such that $\forall \delta\psi \in \mathcal{E}$ and $p_{T} \in \mathcal{D}^{\times}$: $\int\limits_{\mathcal{M}} (A\,\delta\psi)\, p_{T} = \int\limits_{\mathcal{M}} \delta\psi\, (A^{t}\, p_{T})$.
Topological transpose of formal transpose $(A^{t})':\ (\mathcal{D}^{\times})' \to (\mathcal{D}^{\times})'$ is the distributional extension of A. Not always exists.

Fundamental solution on manifolds

Let $E : \boldsymbol{\mathcal{E}} \times \mathcal{D} \to \mathbb{R}$ be Euler-Lagrange functional, and $J \in \mathcal{D}'$.

$\mathtt{K}_{(J)} \in \boldsymbol{\mathcal{E}}$ is solution with source J, iff $\forall \delta\psi_T \in \mathcal{D}:\ (E(\mathtt{K}_{(J)})\,|\,\delta\psi_T) = (J|\delta\psi_T)$.

Specially: one can restrict to $J \in \mathcal{D}^\times \subset \mathcal{E}^\times \subset \mathcal{D}'$.

A continuous map $\mathtt{K} : \mathcal{D}^\times \to \boldsymbol{\mathcal{E}}$ is fundamental solution, iff for all $J \in \mathcal{D}^\times$ the field $\mathtt{K}(J) \in \boldsymbol{\mathcal{E}}$ is solution with source J.

May not exists, and if does, may not be unique.

If $\mathtt{K}_{\psi_0} : \mathcal{D}^\times \to \mathcal{E}$ vectorized fundamental solution is linear (e.g. for linear $E_{\psi_0} : \mathcal{E} \to \mathcal{D}'$):
$\mathtt{K}_{\psi_0} \in \mathcal{L}in(\mathcal{D}^\times, \mathcal{E}) \subset (\mathcal{D}^\times)' \otimes (\mathcal{D}^\times)'$ is distribution.

Particular solutions to the free MDS equation

Distributional solutions to free MDS equation: $G_{\psi_0} = \exp(K_{\psi_0})$ where

$$\begin{aligned} K^{(0)}_{\psi_0} &= 0, \\ K^{(1)}_{\psi_0} &= 0, \\ K^{(2)}_{\psi_0} &= \mathrm{i}\,\hbar\, \mathtt{K}^{(2)}_{\psi_0} \\ K^{(n)}_{\psi_0} &= 0 \qquad\qquad (n \geq 2) \end{aligned}$$

Smooth function solutions to free regularized MDS equation: $G_{\psi_0} = \exp(K_{\psi_0,\kappa})$ where

$$\begin{aligned} K^{(0)}_{\psi_0,\kappa} &= 0, \\ K^{(1)}_{\psi_0,\kappa} &= 0, \\ K^{(2)}_{\psi_0,\kappa} &= \mathrm{i}\,\hbar\, (C_\kappa \otimes C_\kappa)\mathtt{K}^{(2)}_{\psi_0} \\ K^{(n)}_{\psi_0,\kappa} &= 0 \qquad\qquad (n \geq 2) \end{aligned}$$

[Here $C_\kappa(\cdot) := \eta \star (\cdot)$ is convolution by a test function η.]

Renormalization from functional analysis p.o.v.

Let $\mathbb{F}$ and $\mathbb{G}$ real or complex top.vector space, Hausdorff loc.conv complete.

Let $M : \mathbb{F} \rightarrowtail \mathbb{G}$ densely defined linear map (e.g. MDS operator).

Closed: the graph of the map is closed.

Closable: there exists linear extension, such that its graph closed (unique if exists).

Closable $\Leftrightarrow$ where extendable with limits, it is unique.

Multivalued set:
$\mathrm{Mul}(M) := \left\{ y \in \mathbb{G} \,\middle|\, \exists\, (x_n)_{n\in\mathbb{N}} \text{ in } \mathrm{Dom}(M) \text{ such that } \lim_{n\to\infty} x_n = 0 \text{ and } \lim_{n\to\infty} M x_n = y \right\}$.

$\mathrm{Mul}(M)$ always closed subspace.

Closable $\Leftrightarrow \mathrm{Mul}(M) = \{0\}$.

Maximally non-closable $\Leftrightarrow \mathrm{Mul}(M) = \overline{\mathrm{Ran}(M)}$. Pathological, not even closable part.

Polynomial interaction term of MDS operator maximally non-closable!

MDS operator:

$$\mathbf{M}: \quad \mathcal{D}\otimes\mathcal{T}(\mathcal{E})\to\mathcal{T}(\mathcal{E}), \quad G\mapsto \mathbf{M}\,G$$

linear, everywhere defined continuous. So,

$$\mathbf{M}: \quad \mathcal{T}(\mathcal{D}^{\times\prime})\rightarrowtail\mathcal{D}'\otimes\mathcal{T}(\mathcal{D}^{\times\prime}), \quad G\mapsto \mathbf{M}\,G$$

linear, densely defined.

Similarly: $\mathbf{M}_\kappa$ regularized MDS operator (κ: a fix regularizator).

Not good equation:

$$G \in \mathcal{T}(\mathcal{D}^{\times\prime})\,? \qquad G^{(0)} = 1 \text{ and } \exists\, \mathcal{G}_\kappa \to G \text{ approximator sequence, such that :} \quad \lim_{\kappa\to\delta} \mathbf{M}\, \mathcal{G}_\kappa = 0.$$

All G would be selected, because $\mathrm{Mul}()$ set of interaction term is full space.

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Can be good:

$$G \in \mathcal{T}(\mathcal{D}^{\times\prime})\ ? \qquad G^{(0)} = 1 \text{ and } \exists\, \mathcal{G}_\kappa \to G \text{ approximator sequence, such that } : \quad \forall\, \kappa:\ \mathbf{M}_\kappa\, \mathcal{G}_\kappa = 0.$$

That is, as implicit function of κ, not as operator closure kernel.

Running coupling:
If in $\mathbf{M}_\kappa$ EL terms are combined with κ-dependent weights $\gamma(\kappa)$.
(Not just with real factors.)
E.g.:

$$(\gamma, G) \in \mathcal{T}(\mathcal{D}^{\times\prime})\ ? \qquad G^{(0)} = 1 \text{ and } \exists\, \mathcal{G}_\kappa \to G \text{ approximator sequence, such that } : \quad \forall\, \kappa:\ \mathbf{M}_{\gamma(\kappa),\kappa}\, \mathcal{G}_\kappa = 0.$$

Feynman integral "$\Longleftrightarrow$" MDS equation.

Wilsonian regularized Feynman integral:
integrate not on $\mathcal{E}$, only on the image space $C_\kappa[\mathcal{E}]$ of a smoothing operator $C_\kappa : \mathcal{E} \to \mathcal{E}$.

[Smoothing operator: $\sim$ convolution, can be generalized to manifolds. Does UV damping.]
Automatically knows RGE relations.

Wilsonian regularized Feynman integral "$\Longleftrightarrow$" regularized MDS equation + RGE:

$$\left(\psi_0, \kappa \mapsto \gamma(\kappa), \kappa \mapsto \mathcal{G}_{\psi_0,\kappa}\right) = ? \text{ such that : } \qquad \underbrace{\mathcal{G}^{(0)}_{\psi_0,\kappa}}_{=: \, b\,\mathcal{G}_{\psi_0,\kappa}} = 1,$$

$$\forall \kappa : \forall\, \delta\psi_T \in \mathcal{D} : \qquad \underbrace{\left(\iota_{\gamma(\kappa)\,(\mathbf{E}_{\psi_0}\,|\,\delta\psi_T)} - \mathrm{i}\,\hbar\, L_{C_\kappa \delta\psi_T} \right)}_{=: \, \mathbf{M}_{\psi_0,\kappa,\delta\psi_T}} \mathcal{G}_{\psi_0,\kappa} = 0,$$

$$\text{RGE} \longrightarrow \qquad \forall \mu, \kappa : \quad \mathcal{G}^{(n)}_{\psi_0,(C_\mu \kappa)} = \left(\otimes^n C_\mu\right) \mathcal{G}^{(n)}_{\psi_0,\kappa}.$$

Running coupling is meaningful. Conjecture: RG flow of $\mathcal{G}_{\psi_0,\kappa} \leftrightarrow$ distributional G_{ψ_0}.
(Conjecture proved for flat spacetime for bosonic fields.)

Some complications on topological vector spaces

Careful with tensor algebra! Schwartz kernel theorems:

$$\hat{\otimes}_\pi^n \mathcal{E} \equiv \mathcal{E}_n \equiv (\hat{\otimes}_\pi^n \mathcal{E}')' \equiv \mathcal{L}in(\mathcal{E}', \hat{\otimes}_\pi^{n-1} \mathcal{E})$$

$$(\hat{\otimes}_\pi^n \mathcal{E})' \equiv \mathcal{E}'_n \equiv \hat{\otimes}_\pi^n \mathcal{E}' \equiv \mathcal{L}in(\mathcal{E}, \hat{\otimes}_\pi^{n-1} \mathcal{E}')$$

$$\hat{\otimes}_\pi^n \mathcal{D} \leftarrow \mathcal{D}_n \equiv (\hat{\otimes}_\pi^n \mathcal{D}')'$$

cont.bij.

$$(\hat{\otimes}_\pi^n \mathcal{D})' \rightarrow \mathcal{D}'_n \equiv \hat{\otimes}_\pi^n \mathcal{D}' \equiv \mathcal{L}in(\mathcal{D}, \hat{\otimes}_\pi^{n-1} \mathcal{D}')$$

$\mathcal{E} \times \mathcal{E} \to F$ separately continuous maps are jointly continuous.
$\mathcal{E}' \times \mathcal{E}' \to F$ separately continuous bilinear maps are jointly continuous.
For mixed, no guarantee.
For $\mathcal{D}$ or $\mathcal{D}'$ spaces, joint continuity from separate continuity of bilinear forms not automatic.
For mixed, even less guarantee.

But as convergence vector spaces, everything is nice with mixed $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{D}$, $\mathcal{D}'$ multilinears (separate sequential continuity $\Leftrightarrow$ joint sequential continuity).