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Causal boundary for strongly causal spacetimes

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Abstract. A study of the general framework for the causal boundary is presented, and for various separation properties necessary and sufficient conditions are proved. As a consequence, it is shown that the usual implicit topological identification rule, imposed on the 'preboundary' of the spacetime to obtain the right boundary structure, might in general not exist. To rule out this difficulty, a new explicit identification rule is proposed. This new identification seems to yield the intuitively expected point set structure of the boundary, onto which the chronology relation can be extended.

1. Introduction

In the general theory of relativity the spacetime model is a smooth manifold M without a boundary equipped with a Lorentzian metric. To study some special problems, e.g. asymptotic structure, singularities, etc, however, the introduction of certain kinds of boundary points seems to be useful. (To prove singularity theorems predicting curvature singularities, a boundary construction is probably indispensable [1].) Several attempts have been made to construct a boundary to spacetime, among which the causal boundary is probably the simplest and most transparent one [2]. Its basic idea is that the future (past) inextendible non-spacelike curves having the same chronological past (future) define a point at a future (past) singularity or infinity. Certain boundary points defined by past inextendible non-spacelike curves, however, have to be identified with certain boundary points defined by future endless curves.

The original identification rule of Geroch *et al* [2] for strongly causal spacetimes is given in an implicit way. Thus it is rather complicated to construct the causal boundary for a given spacetime, and for Taub's spacetime this identification seems to conflict with the intuitively expected boundary structure [3].

Budic and Sachs proposed an explicit identification rule for causally continuous spacetimes [4]. For Taub's spacetime their identification yields a more reasonable boundary structure. Unfortunately, their method cannot be used for strongly causal spacetimes and, in general, there is no known explicit rule to carry out this identification.

The present paper is devoted to the causal boundary too. In § 2 the general framework for the causal boundary is examined, where we concentrate on the separation properties of quotient spaces. Based on these results, in § 3 we examine some of the properties of the topology proposed by Geroch *et al* and it is shown that the original implicit topological identification rule, given for strongly causal spacetimes, might not exist in general. Thus we have the twofold problem of finding an appropriate topology and an identification rule. In the remaining part of the paper a new explicit identification and some of its consequences are described. Though we could not give any

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reasonable topology on the completed spacetime, satisfying all our requirements, the point set structure of the boundary seems to be the expected one; moreover, the chronology relation can be extended to the boundary in a natural way.

Throughout this paper spacetime is assumed to be time-oriented and strongly causal [5]. Our conventions are the same as those used by Hawking and Ellis [6], unless otherwise stated. The future and past sets are assumed to be open [5].

2. General framework for causal boundary

To introduce some kind of boundary for the spacetime (M, g) one has to have a topological space $(\bar{M}, \bar{\mathscr{T}})$ and an embedding Φ: $(M, \mathscr{T}) \to (\bar{M}, \bar{\mathscr{T}})$ such that $\Phi(M) \subset \bar{M}$ is an open dense subspace and the set $\partial := \bar{M} - \Phi(M)$ is interpreted as the boundary of M [7]. ($\mathscr{T}$ is the Alexandrov or manifold topology, which coincide in strongly causal spacetimes [5].)

By causal boundary we mean a boundary construction using only conformally invariant concepts, TIP and TIF [2] to represent the boundary points themselves. If M^+ and M^- denote the collection of IF and IP, respectively, then, because of the distinguishing conditions [8], the mappings $I^{\pm}: M \to M^{\pm}: p \mapsto I^{\pm}(p)$ are injective. On the disjoint union $M^+ \cup M^-$ one can define the equivalence relation $\mathscr{R}_0$ that identifies PIP with the corresponding PIF: for $F \in M^+$, $P \in M^-(P, F) \in \mathscr{R}_0$ if $P = I^-(p)$ and $F = I^+(p)$ for some $p \in M$ (i.e. $P = (I^- \circ (I^+)^{-1})(F)$) and for $\forall F \in M^+$, $\forall P \in M^-(F, F)$, $(P, P) \in \mathscr{R}_0$. If $M^{\#} := M^+ \cup M^- / \mathscr{R}_0$, then the map $i: M \to M^{\#}: p \mapsto i(p)$ is injective, where $i(p)$ stands for the identified pair $(I^+(p), I^-(p))$; furthermore, $M^{\#}$ is the disjoint union of $i(M)$, the collection ∂^- of TIP ('future preboundary') and the collection ∂^+, ∂^- of TIP ('past preboundary'). Thus one can think of $M^{\#}$ as the spacetime with additional 'preboundary' points. However, if we want to recover the boundary structure of certain well known simple spacetimes obtained earlier by the conformal technique [9], then certain preboundary points in $M^{\#}$ have to be identified. Therefore we need a further identification on $M^{\#}$, which is an equivalence relation $\mathscr{R}$ on $M^{\#}$ such that for each pair $(A, B) \in \mathscr{R}$ $A \neq B$ implies A, $B \in \partial^+ \cup \partial^-$, i.e. $\mathscr{R}$ is trivial on $i(M)$. (The identification of inner and preboundary points cannot be allowed, as otherwise originally inextendible non-spacelike curves would become curves having endpoints in M. Thus, in constructing the boundary, the structure of M itself would change.) The completed spacetime is $\bar{M} := M^{\#}/\mathscr{R}$ and $\partial_c := \bar{M} - i(M)$ is the causal boundary for M. The points $b \in \partial_c$ can be considered as endpoints of inextendible non-spacelike curves.

The minimal requirement for the topology $\bar{\mathscr{T}}$, beyond the ones mentioned above, is the T_1 separation of inner and boundary points [10]. In every mathematical model where more than one structure exist together, the cooperation, or rather the compatibility, of these structures is expected. In the present case two notions of endpoints of the non-spacelike curves, which are endless in M, are introduced. The first one is given by causality: it is the $\mathscr{R}$ equivalence class determined by the corresponding TIP or TIF. The second one is defined by the topology: a point x is said to be an endpoint of the curve $\gamma(t)$ if for every neighbourhood U of x there exists a parameter value t_0 such that for $\forall t > t_0$, $\gamma(t) \in U$ [6, 9, 10]. Thus it is expected that the causal endpoints of inextendible non-spacelike curves be topological endpoints too. The requirement of the uniqueness of the endpoints of (not only inextendible) non-spacelike curves can be considered as a separation axiom for $\bar{\mathscr{T}}$ between the axioms T_1 and T_2: in a Hausdorff space each non-spacelike curve has a unique past and future endpoint, and if every $\bar{\mathscr{T}}$ neighbourhood of x contained a point $y \neq x$, then x would also be an endpoint of all the curves for which y is an endpoint.

Inextendible spacelike curves may not have any endpoints, or may have more than one endpoint. Thus, outside of mathematical convenience, we cannot see any reason to require the Hausdorffness of the space. In fact, the occurrence of certain non-Hausdorff separated points is inevitable in the NUT extension of Taub's spacetime [6, 10–13]. The uniqueness of the endpoints of non-spacelike curves, however, seems to be a reasonable 'physical' separation axiom which, thus, can be expected to hold.

One can easily show that no generality is lost if the topology $\bar{\mathscr{T}}$ is considered as the quotient topology $\mathscr{T}^{\#}/\mathscr{R}$ of an appropriate $\mathscr{T}^{\#}$ given on $M^{\#}$. Following Geroch *et al* [2], we too prefer this way and in the rest of this section some of the general properties of a quotient topology will be considered.

Let us start with a topology $\mathscr{T}^{\#}$ on $M^{\#}$, for which i: $(M, \mathscr{T}) \to (M^{\#}, \mathscr{T}^{\#})$ is an open dense embedding and, for every inextendible non-spacelike curve γ, $P := I^-[\gamma]$ and $F := I^+[\gamma]$ are future and past endpoints of $i \circ \gamma$, respectively.

Proposition 2.1. If $\mathscr{R}$ is any equivalence relation, being trivial on $i(M)$, and $\pi : M^{\#} \to M^{\#}/\mathscr{R}$ is the corresponding canonical projection, then $\pi \circ i : (M, \mathscr{T}) \to (M^{\#}/\mathscr{R}, \mathscr{T}^{\#}/\mathscr{R})$ is an open dense embedding. Moreover $\pi(P)$ and $\pi(F)$ are future and past endpoints of $\pi \circ i \circ \gamma$, respectively.

Proof. i is injective and $\mathscr{R}$ is trivial on $i(M)$. Thus $\pi \circ i$ is injective too. Both i and π are continuous and hence $\pi \circ i$ is also continuous. To prove that $\pi \circ i$ is open, let U be a $\mathscr{T}$-open subset of M. Then $i(U)$ is $\mathscr{T}^{\#}$-open and $i(U) = \pi^{-1}((\pi \circ i)(U))$, because π is injective on $i(M)$. Therefore the set $V := (\pi \circ i)(U)$ has an open preimage in $M^{\#}$ by the continuous map π, i.e. V is open in $\mathscr{T}^{\#}/\mathscr{R}$. In particular $(\pi \circ i)(M)$ is also open. If W is a $\mathscr{T}^{\#}/\mathscr{R}$-open neighbourhood of $b \in \partial^+ \cup \partial^- / \mathscr{R}$ then $\pi^{-1}(W)$ is a $\mathscr{T}^{\#}$-open neighbourhood of $\forall B \in \pi^{-1}(b)$. $i(M)$ is dense in $(M^{\#}, \mathscr{T}^{\#})$ and thus $\pi^{-1}(W) \cap i(M) \neq \varnothing$, i.e. $W \cap i(M) \neq \varnothing$. Therefore $\pi \circ i$ is an open dense embedding.

If $P := I^-[\gamma]$ is a future endpoint of $i \circ \gamma$, then let V be a $\mathscr{T}^{\#}/\mathscr{R}$-open neighbourhood of $\pi(P)$. Then $\pi^{-1}(V)$ is $\mathscr{T}^{\#}$-open and $P \in \pi^{-1}(V)$. Thus for some parameter value t_0 and for $\forall t > t_0$, $i \circ \gamma(t) \in \pi^{-1}(V)$. This, however, implies that $\pi \circ i \circ \gamma(t) \in V$, i.e. $\pi(P)$ is a future endpoint of $\pi \circ i \circ \gamma$. Similarly, $\pi(F)$ is a past endpoint of $\pi \circ i \circ \gamma$, or simply $i \circ \gamma$.

As a corollary to proposition 2.1 each point of $M^{\#}/\mathscr{R}$ is a past or future endpoint of some non-spacelike curve. Furthermore each such curve has (topological) past and future endpoints. These endpoints, however, are not necessarily unique; their uniqueness depends on the separation properties of the space. First, the separation of inner and boundary points is considered.

Proposition 2.2. (a) The boundary points are T_1 separated from the inner points in the topology $\mathscr{T}^{\#}/\mathscr{R}$ iff all the single point sets $\{i(q)\}$, $q \in M$, are closed in $\mathscr{T}^{\#}$.

(b) If the points $b \in \partial^+ \cup \partial^- / \mathscr{R}$, $i(q)(q \in M)$ are T_2 separated in $\mathscr{T}^{\#}/\mathscr{R}$, then $\forall B \in \pi^{-1}(b)$ are T_2 separated from $i(q)$ in $\mathscr{T}^{\#}$.

(c) If the point $p \in M$ has a $\mathscr{T}$-open neighbourhood V such that each $B \in \partial^+ \cup \partial^-$ has a $\mathscr{T}^{\#}$-open neighbourhood U_B for which $U_B \cap i(V) = \varnothing$, then all the boundary points $b \in \partial^+ \cup \partial^- / \mathcal{R}$ are T_2 separated from $i(p)$ in $\mathcal{T}^\# / \mathcal{R}$. (The neighbourhood V above will be called a $\mathcal{T}^\#$-universal neighbourhood of p.)

Proof. (a) Inner points are T_2 separated in $\mathcal{T}^\# / \mathcal{R}$. Thus, if for $\forall q \in M$, $i(q)$ is T_1 separated from the boundary points in $\mathcal{T}^\# / \mathcal{R}$, then $\{i(q)\}$ is closed in $\mathcal{T}^\# / \mathcal{R}$ and thus in $\mathcal{T}^\#$ too. Conversely, for $\forall b \in \partial^+ \cup \partial^- / \mathcal{R}$ and $\forall q \in M$, $i(M)$ is a $\mathcal{T}^\# / \mathcal{R}$-open neighbourhood of $i(q)$, not containing b. If $\{i(q)\}$ is closed in $\mathcal{T}^\#$, then $M^\# - \{i(q)\} = \pi^{-1}(\pi(M^\# - \{i(q)\}))$ is open in $\mathcal{T}^\#$. Thus $M^\# / \mathcal{R} - \{i(q)\} = \pi(M^\# - \{i(q)\})$ is a $\mathcal{T}^\# / \mathcal{R}$-open neighbourhood of $\forall b$.

(b) If W and U are disjoint $\mathcal{T}^\# / \mathcal{R}$-open neighbourhoods of b and $i(p)$, respectively, then $\pi^{-1}(W)$ and $\pi^{-1}(V)$ are disjoint $\mathcal{T}^\#$-open neighbourhoods of all the $B \in \pi^{-1}(b)$ and $i(p)$, respectively.

(c) If

$$U := \bigcup_{B \in \partial^+ \cup \partial^-} U_B \qquad W := \pi(U)$$

then $b \in W$ and, as a consequence of $U \cap i(M) = W \cap i(M)$, $W \cap i(V) = \emptyset$. But W is $\mathcal{T}^\# / \mathcal{R}$-open, since $\pi^{-1}(W) = \partial^+ \cup \partial^- \cup (U \cap i(M)) = U$ is open in $\mathcal{T}^\#$.

In terms of the topology $\mathcal{T}^\#$, proposition 2.2 gives an equivalent condition for the T_1 separation, a necessary condition, and a sufficient condition for the T_2 separation of inner and boundary points, respectively. Thus if each inner point has a $\mathcal{T}^\#$-universal neighbourhood, then for any non-spacelike curve γ with endpoints in M the endpoints of $i \circ \gamma$ in $(M^\# / \mathcal{R}, \mathcal{T}^\# / \mathcal{R})$ are unique.

The topology $\mathcal{T}^\#$ is expected to be defined in terms of causality, and now the notion of causal topology will be defined. A set U is called causal if no non-spacelike curve leaving U can re-enter U; and the topology $\mathcal{T}^\#$ is said to be causal if every point has a neighbourhood base consisting of causal open sets. The next proposition gives equivalent statements for two separation properties of the entire space $(M^\# / \mathcal{R}, \mathcal{T}^\# / \mathcal{R})$.

Proposition 2.3. (a) The space $(M^\# / \mathcal{R}, \mathcal{T}^\# / \mathcal{R})$ is a T_1 space iff the set $[X] := \{Z \in M^\# \mid (X, Z) \in \mathcal{R}\}$ is closed in $\mathcal{T}^\#$ for $\forall X \in M^\#$.

(b) Let $\mathcal{T}^\#$ be a causal topology such that each point of M has a $\mathcal{T}^\#$-universal neighbourhood; let γ be a non-spacelike curve, $P := I^-[\gamma]$ and $b := \pi(P)$. b is the unique future endpoint of $i \circ \gamma$ in $(M^\# / \mathcal{R}, \mathcal{T}^\# / \mathcal{R})$ iff $[P]$ is closed in $\mathcal{T}^\#$ and every future endpoint of $i \circ \gamma$ in $(M^\#, \mathcal{T}^\#)$ are $\mathcal{R}$-equivalent to P.

Proof. (a) Let $x \in M^\# / \mathcal{R}$ and $X \in \pi^{-1}(x)$. If the space $(M^\# / \mathcal{R}, \mathcal{T}^\# / \mathcal{R})$ is T_1 then $\{x\}$ is closed in $\mathcal{T}^\# / \mathcal{R}$. Thus $[X] = \{Z \in \pi^{-1}(x)\} = \pi^{-1}(\{x\})$ is closed in $\mathcal{T}^\#$. If $[X]$ is closed in $\mathcal{T}^\#$ then $M^\# - [X]$ is open. Moreover $M^\# - [X] = \pi^{-1}(\pi(M^\# - [X]))$. Thus $M^\# / \mathcal{R} - \{x\} = \pi(M^\# - [X])$ is open, i.e. $\{x\}$ is closed in $\mathcal{T}^\# / \mathcal{R}$. For $\forall X \in M^\#$ this, however, implies that the quotient space is T_1.

(b) Let b be the unique future endpoint of $i \circ \gamma$ in the quotient space. If B is a future endpoint of $i \circ \gamma$ in $(M^\#, \mathcal{T}^\#)$, then $\pi(B)$ is an endpoint of $i \circ \gamma$ in the quotient space too. Thus B must be $\mathcal{R}$-equivalent to P. Since b is unique, $[P]$ is closed in $\mathcal{T}^\#$.

Let $[P]$ be closed in $\mathcal{T}^\#$ and all the future endpoints of $i \circ \gamma$ in $\mathcal{T}^\#$ be $\mathcal{R}$-equivalent to P. Let $[0, 1)$ be the parameter domain of γ. First we show that for $\forall \delta \in (0, 1)$ the portion $i \circ \gamma([0, \delta])$ is closed in $\mathcal{T}^\#$. $\gamma([0, \delta])$ is closed in the manifold topology $\mathcal{T}$.

Thus for $\forall p \in M - \gamma([0, \delta])$ the set $i(M - \gamma([0, \delta]))$ is a $\mathscr{T}^{\#}$-open neighbourhood of $i(p)$, which does not intersect $i \circ \gamma([0, \delta])$. $\gamma([0, \delta])$ is compact in the manifold topology and so it can be covered by finitely many $\mathscr{T}^{\#}$-universal neighbourhoods: $\gamma([0, \delta]) \subset V_1 \cup \ldots \cup V_r$. If $B \in \partial^+ \cup \partial^-$ and $U_{B1}, \ldots, U_{Br}$ are the $\mathscr{T}^{\#}$-open neighbourhoods of B having an empty intersection with $i(V_1), \ldots, i(V_r)$, respectively, then $U := U_{B1} \cap \ldots \cap U_{Br}$ is a $\mathscr{T}^{\#}$-open neighbourhood of B having an empty intersection with $i \circ \gamma([0, \delta])$. Thus $i \circ \gamma([0, \delta])$ is closed in $\mathscr{T}^{\#}$.

Let $B \in \partial^+ \cup \partial^- - [P]$. $\mathscr{T}^{\#}$ is a causal topology and B is not an endpoint of $i \circ \gamma$ in $\mathscr{T}^{\#}$. Thus B has a $\mathscr{T}^{\#}$-open neighbourhood $\tilde{U}_B$ and $\exists \delta \in (0, 1)$ such that $\tilde{U}_B \cap i \circ \gamma([\delta, 1)) = \varnothing$. $[P]$ and $i \circ \gamma([0, \delta])$ are closed in $\mathscr{T}^{\#}$ and thus $U_B := \tilde{U}_B - [P] - i \circ \gamma([0, \delta])$ is open too. Define

$$U := \bigcup_{B \in \partial^+ \cup \partial^- - [P]} U_B.$$

Then $\pi(U) \cap i \circ \gamma = \varnothing$, $b' \in \pi(U)$ for $\forall b' \in \partial^+ \cup \partial^- / \mathscr{R} - \{b\}$ and $\pi^{-1}(\pi(U)) = \pi^{-1}(\cup \pi(U_B)) = \pi^{-1}((\partial^+ \cup \partial^- / \mathscr{R} - \{b\}) \cup (U \cap i(M)) = (\partial^+ \cup \partial^- - [P]) \cup (U \cap i(M)) = U$, i.e. $\pi(U)$ is a $\mathscr{T}^{\#}/\mathscr{R}$-open neighbourhood of every boundary point different from b, into which $i \circ \gamma$ does not enter. Thus b is unique.

The main problem is therefore to determine which points of $\partial^+ \cup \partial^-$ should be identified and how should the topology be defined.

3. The topology $\mathscr{T}^{\#}$ and the GKP identification rule

A candidate for $\mathscr{T}^{\#}$ is the topology proposed by Geroch *et al* [2]. For each non-empty irreducible future set F they define the sets

$$F^{\text{int}} := \{P \in i(M) \cup \partial^+ \mid P \cap F \neq \varnothing\}$$

$$F^{\text{ext}} := \{P \in i(M) \cup \partial^+ \mid \text{if } P = I^-[S] \text{ for some } S \subset M, \text{ then } I^+[S] \not\subset F\}$$

and similarly P^{int}, P^{ext} for each irreducible past set P. Naturally, $F^{\text{int}} \cap F^{\text{ext}} = \varnothing$ and the collection $\mathscr{C}^{\#} := \{P^{\text{int}}, P^{\text{ext}}, F^{\text{int}}, F^{\text{ext}} \mid P \in M^-, F \in M^+\}$ is a covering of $M^{\#}$. Thus there is a coarse topology $\mathscr{T}^{\#}_{\text{GKP}}$, or in this section simply $\mathscr{T}^{\#}$, on $M^{\#}$ for which $\mathscr{C}^{\#}$ is a subbase [2].

Proposition 3.1. (*a*) The map $i : (M, \mathscr{T}) \to (M^{\#}, \mathscr{T}^{\#})$ is an open dense embedding.

(*b*) If γ is a future endless non-spacelike curve in M, then $P := I^-[\gamma] \in \partial^+$ is a future endpoint of $i \circ \gamma$ in the topology $\mathscr{T}^{\#}$.

(*c*) $\mathscr{T}^{\#}$ is a causal topology.

Proof. (*a*) Due to the distinguishing conditions, i is injective. i is continuous, because $i^{-1}(F^{\text{int}}) = F$, $i^{-1}(F^{\text{ext}}) = M - \bar{F}$ for $\forall F \in M^+$ and similarly $i^{-1}(P^{\text{int}}) = P$, $i^{-1}(P^{\text{ext}}) = M - \bar{P}$, where the bar denotes closure in the manifold topology. For $\forall p \in M$, $i(I^+(p)) = (I^+(p))^{\text{int}} \cap i(M)$. But $(I^+(p))^{\text{int}} \subset i(M) \cup \partial^+$ and the collection $\{P^{\text{int}} \mid P \in \partial^+\}$ constitutes a covering of $i(M) \cup \partial^-$. Thus $i(I^+(p)) = \cup\{(I^+(p))^{\text{int}} \cap P^{\text{int}} \mid P \in \partial^+\}$, which is open in $\mathscr{T}^{\#}$. Similarly $i(I^-(p))$ is also open in $\mathscr{T}^{\#}$. Thus i is an embedding. $i(M)$ is open in $\mathscr{T}^{\#}$ because $i(M) = \cup\{i(I^+(p) \cap I^-(q)) \mid p \ll q\}$. If U is a $\mathscr{T}^{\#}$-open neighbourhood of $P \in \partial^+$ then for some point $p \in P$ and IF $F_1, \ldots, F_n$, $P \in (I^+(p))^{\text{int}} \cap F_1^{\text{ext}} \cap \ldots \cap F_n^{\text{ext}} \subset U$. Then $P \cap I^+(p) \neq \varnothing$ and $P \subset (M - \bar{F}_1) \cap \ldots \cap (M - \bar{F}_n)$ and there is a set $S \subset M$ such that $P = I^-[S]$ and $S \subset I^+(p) \cap (M - \bar{F}_1) \cap \ldots \cap (M - \bar{F}_n) = i^{-1}((I^+(p))^{\text{int}} \cap F_1^{\text{ext}} \cap \ldots \cap F_n^{\text{ext}}) \subset i^{-1}(U)$, i.e. $i(M)$ is dense in $(M^\#, \mathcal{T}^\#)$.

(b) If U is a $\mathcal{T}^\#$-open neighbourhood of P then for some $p \in P$ and IF $F_1, \ldots, F_n$, $P \in (I^+(p))^{\text{int}} \cap F_1^{\text{ext}} \cap \ldots \cap F_n^{\text{ext}} \subset U$. $\gamma: [0, 1) \to M$ is a generator of P and thus for sufficiently small $\varepsilon > 0$ and final segment $S_\varepsilon := \gamma([1-\varepsilon, 1))$ one has $P = I^-[S_\varepsilon]$. $P \in F_1^{\text{ext}} \cap \ldots \cap F_n^{\text{ext}}$ implies $I^+[S_\varepsilon] \not\subset F_1 \cup \ldots \cup F_n$ from which $s \notin \bar{F}_1 \cup \ldots \cup \bar{F}_n$ follows for $\forall s \in S_\varepsilon$. Therefore there is a parameter value t_0 such that for $\forall t > t_0$, $\gamma(t) \in I^+(p) \cap (M - \bar{F}_1) \cap \ldots \cap (M - \bar{F}_n)$. This, however, implies $i \circ \gamma(t) \in U$, i.e. P is a future endpoint of $i \circ \gamma$.

(c) If $P \in i(M) \cup \partial^+$ then the sets of the form $(I^+(p))^{\text{int}} \cap F_1^{\text{ext}} \cap \ldots \cap F_r^{\text{ext}}$, $p \in P$, constitute a $\mathcal{T}^\#$-neighbourhood base of P. These neighbourhoods, however, are causal sets.

In the rest of this section the separation properties of $\mathcal{T}^\#$ will be considered.

Since i is an embedding, any two different points of $i(M) \subset M^\#$ (inner points) are T_2 separated in $\mathcal{T}^\#$.

Let $q \in M$ and $P \in \partial^+$. If $q \in \bar{P}$ then let V be a causally convex $\mathcal{T}$-open neighbourhood of q with compact closure. For all $r \in V \cap P$ there is a timelike generator γ of P starting at r. γ is future endless, so it has to leave V and cannot re-enter V [5, 6]. Thus $\exists p \in \gamma$ such that $I^+(p) \cap V = \emptyset$. Therefore $i(V)$ and $(I^+(p))^{\text{int}}$ are disjoint $\mathcal{T}^\#$-open neighbourhoods of $i(q)$ and P, respectively. If $q \in \uparrow P$ then there is an IF F such that $q \in F \subset \uparrow P$ ($\uparrow P$ is the chronological common future of P [14]). Then $i(q) \in F^{\text{int}}$ and if F^{ext} did not contain P then there would be a set $S \subset M$ such that $P = I^-[S]$ and $I^+[S] \subset F$. But then $S \subset \bar{P} \cap \bar{F}$ would hold which, however, would imply that the strong causality condition is violated at the points of S. If $q \in M - \bar{P} - \overline{\uparrow P}$ then there exists a causally convex neighbourhood $V = I^+(u) \cap I^-(v)$ of q and a point $p \in P$ such that $V \cap \bar{P} = \emptyset$ and $V \cap I^+(p) = \emptyset$. $(I^+(p))^{\text{int}}$ and $i(V)$ are disjoint $\mathcal{T}^\#$-open neighbourhoods of P and $i(q)$, respectively. Finally, if $q \in \partial\uparrow P$ then $I^+(q) \subset \uparrow P$. Thus $P \in (I^+(q))^{\text{ext}}$, $i(q) \notin (I^+(q))^{\text{ext}}$ and $i(q) \in i(M)$, $P \notin i(M)$, i.e. P and $i(q)$ are T_1 separated in $\mathcal{T}^\#$. Thus we have proved the next statement.

Proposition 3.2. For all $q \in M$ and $B \in \partial^+ \cup \partial^-$ the points $i(q)$ and B are T_1 separated in $\mathcal{T}^\#$.

A simple consequence of propositions 2.2(a) and 3.2 is that the inner and boundary points of the quotient space $(M^\#/\mathcal{R}, \mathcal{T}^\#/\mathcal{R})$ are T_1 separated for any identification $\mathcal{R}$.

In general, however, inner and preboundary points of $(M^\#, \mathcal{T}^\#)$ are not T_2 separated; moreover, for non-spacelike curves γ with endpoint q in $(M, \mathcal{T})$ the endpoint $i(q)$ of $i \circ \gamma$ is not necessarily unique in $(M^\#, \mathcal{T}^\#)$ either. Figure 1 shows a two-dimensional spacetime in which the TIP P and the point $i(q)$, $q \in \partial\uparrow P$, are T_1 separated in $(M^\#, \mathcal{T}^\#)$; but the TIP P is a future endpoint of the timelike curves ending at q, and thus P and $i(q)$ are not T_2 separated. This spacetime is obtained from that shown by figure 37 of [6], cutting out the countable many closed segments $L_0, L_1, L_2, \ldots$. This spacetime satisfies Carter's nth-order strong causality condition [15] for $\forall n \in \mathbb{N}$ (i.e. the '∞th' strong causality condition) but is not stable causal [6].

Proposition 2.2(b) gives us a necessary condition for the existence of an identification $\mathcal{R}_H$ yielding T_2 quotient topology: inner and preboundary points of $M^\#$ must be T_2 separated in $\mathcal{T}^\#$. Thus in general strongly causal spacetimes an identification gluing together only preboundary points and yielding a Hausdorff quotient topology

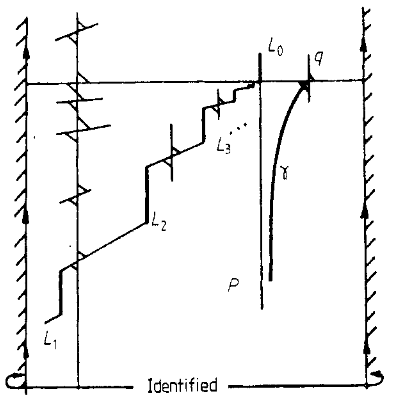

Figure 1. The TIP P is an endpoint of all the timelike curves $i \circ \gamma$ in $\mathcal{T}^{\#}_{\mathrm{GKP}}$, for which $q \in \partial\uparrow P$ is an endpoint in $\mathcal{T}$.

might not exist. Geroch *et al* [2] defined $\mathcal{R}$ as the minimal identification yielding T_2 quotient space or, more precisely, $\mathcal{R}_{\mathrm{GKP}}$ is the intersection of all the equivalence relations $\mathcal{R}_H$ above [2]. Consequently, unfortunately the identification rule of Geroch *et al* might not exist in general. For the uniqueness of the $\mathcal{T}^{\#}/\mathcal{R}$ endpoints of a non-spacelike curve in the quotient space the $\mathcal{R}$ equivalence of the $\mathcal{T}^{\#}$ endpoints in $M^{\#}$ is necessary (see the proof of proposition 2.3(b)). Thus an identification $\mathcal{R}$ gluing together only preboundary points and yielding unique endpoints for world lines in $(M^{\#}/\mathcal{R}, \mathcal{T}^{\#}/\mathcal{R})$ might not exist either.

In stable causal spacetimes, however, such situations cannot occur, as follows from proposition 2.2(c) and 3.3 below.

Proposition 3.3. If M is stable causal then every point $p \in M$ has a $\mathcal{T}^{\#}$-universal neighbourhood.

Proof. Suppose, on the contrary, that for some point $p \in M$ and every $\mathcal{T}$-open neighbourhood $\langle a_n, b_n\rangle$ of p there is a preboundary point B_n with no $\mathcal{T}^{\#}$-open neighbourhood having an empty intersection with $i(\langle a_n, b_n\rangle)$. ($\langle a_n, b_n\rangle$ is the chronological interval $I^+(a_n) \cap I^-(b_n)$.) Let B_n be a TIP and $B_n = I^-[S'_n]$ for some $S'_n \subset M$. But $I^-[S'_n] \not\subset I^-(b_n)$ implies the existence of a point $r_n \in I^-[S'_n] - I^-(b_n)$, from which $B_n \in (I^+(r_n))^{\mathrm{int}}$ and $(I^+(r_n))^{\mathrm{int}} \cap i(\langle a_n, b_n\rangle) = \varnothing$ follow. Thus, according to our hypothesis, $I^-[S'_n] \subset I^-(b_n)$ must hold. If for every set S satisfying $I^-[S'_n] = I^-[S]$ $I^+[S] \not\subset I^+(a_n)$ held, then $B_n \in (I^+(a_n))^{\mathrm{ext}}$ and $(I^+(a_n))^{\mathrm{ext}} \cap i(\langle a_n, b_n\rangle) = \varnothing$ would hold. Using a similar argument for the case B_n being a TIF, it follows that for every $\langle a_n, b_n\rangle$ there is a set $S_n \subset M$ such that both $I^+[S_n] \subset I^+(a_n)$, $I^-[S_n] \subset I^-(b_n)$ hold and at least one of the sets $I^+[S_n]$, $I^-[S_n]$ is terminal and indecomposable (see [2]). If $\langle u, v\rangle$ is a $\mathcal{T}$-open neighbourhood of p with a compact closure, then, without loss of any generality, one can assume that $S_n \cap \langle u, v\rangle = \varnothing$.

Let t be the global time function on M and $\varepsilon > 0$. Let $\{a_n\}$ and $\{b_n\}$ converge to p, let s_n be a point of S_n and W_n a $\mathcal{T}$-open neighbourhood of s_n. There are points $x_n \in I^-(s_n) \cap W_n$ and $y_n \in I^+(s_n) \cap W_n$ such that

$$|t(s_n) - t(x_n)| < \varepsilon/4n \qquad |t(y_n) - t(s_n)| < \varepsilon/4n.$$

Since $I^+(s_n) \subset I^+[S_n] \subset I^+(a_n)$ and $I^-(s_n) \subset I^-(b_n)$, the causal relations $x_n \ll b_n$, and

$a_n \ll y_n$ follow, implying that

$$t(a_n) - \varepsilon/4n < t(y_n) - \varepsilon/4n < t(s_n) < t(x_n) + \varepsilon/4n < t(b_n) + \varepsilon/4n.$$

t is continuous and $\{a_n\}$, $\{b_n\} \to p$. Thus

$$t(a_n),\ t(y_n),\ t(s_n),\ t(x_n),\ t(b_n) \to t(p)$$

i.e. $\exists n_0 \in \mathbb{N}$ such that for $\forall n > n_0$

$$|t(p) - t(a_n)| < \tfrac{1}{2}\varepsilon \qquad |t(a_n) - t(y_n)| < \tfrac{1}{2}\varepsilon \qquad |t(y_n) - t(s_n)| < \tfrac{1}{2}\varepsilon.$$

For $\forall \delta > 0$ let $U_\delta := \{r \in \langle u, v\rangle | t(p) - \delta < t(r) < t(p) + \delta\}$. Let λ be any future directed timelike curve starting at a point $a \in I^-(p) \cap U_{\varepsilon/2}$, and let μ be any past directed timelike curve starting at a point $y \in \lambda \cap U_{\varepsilon/2}$. Then for sufficiently small ε the increment of the time function t along the segment of every such λ lying in $U_{\varepsilon/2}$ is not less than $\frac{1}{2}\varepsilon$. Furthermore the decrement of t along the segment of these μ lying in U_ε is not less than $\frac{1}{2}\varepsilon$. Hence the causal relations $a_n \ll p$, $a_n \ll y_n$, $s_n \ll y_n$ and the corresponding convergences imply that $s_n \in U_\varepsilon \subset \langle u, v\rangle$, which, however, contradicts $S_n \cap \langle u, v\rangle = \emptyset$.

For the sake of completeness consider the separation of preboundary points and let P, $P' \in \partial^+$ and $F \in \partial^-$. If $F \not\subset \uparrow P$, then $\exists q \in F$ such that $P - \overline{I^-(q)} \neq \emptyset$ and let $p \in P - \overline{I^-(q)}$. Then $(I^+(p))^{\rm int}$ and $(I^-(q))^{\rm int}$ are disjoint $\mathscr{T}^{\#}$-open neighbourhoods of P and F, respectively. If γ is a generator of P and $F \subset \uparrow P$, then $\gamma \subset \bar{P}$ and $F \in P^{\rm ext}$. Thus F cannot be an endpoint of $i \circ \gamma$. If $P' \not\subset P$ and γ is a generator of P then for every point $p' \in P' - \bar{P}\, P' \in (I^+(p'))^{\rm int}$ and $\bar{P} \cap I^+(p') = \emptyset$, i.e. P' cannot be an endpoint of $i \circ \gamma$.

If $P \not\subset P' \subset P$ then, however, P and P' are not necessarily T_1 separated even if M is stable causal. Figure 2 shows a two-dimensional spacetime, obtained from the quarter Minkowski spacetime by cutting out the countable many closed segments $L_1, L_2, \ldots,$ in which the TIP P and P' are not T_1 separated in $\mathscr{T}^{\#}$. One can also find stable causal spacetimes in which neither the TIP nor the TIF are T_2 separated.

If, however, M is causally continuous, then each non-spacelike curve has a unique past and future endpoint in $(M^{\#}, \mathscr{T}^{\#})$ and if M is globally hyperbolic then the space $(M^{\#}, \mathscr{T}^{\#})$ is T_2.

Finally, one can therefore conclude that either or both the topology $\mathscr{T}^{\#}_{\rm GKP}$ and the identification rule $\mathscr{R}_{\rm GKP}$ have to be abandoned. Though a great variety of modified forms of $\mathscr{T}^{\#}_{\rm GKP}$, for which a statement like proposition 3.1 holds, can be introduced, but all these have a pathology like that is shown by figure 1. The 'violation of strong causality condition at the preboundary point P' prevents the T_2 separation of P and

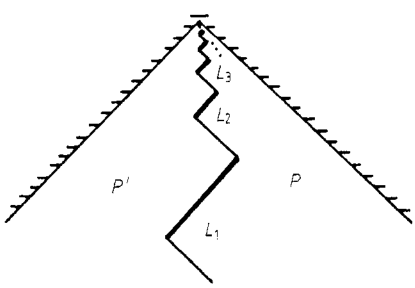

Figure 2. The TIP P, P' are T_0 but not T_1 separated in $\mathscr{T}^{\#}_{\rm GKP}$.

$i(q)$, $q \in \partial\uparrow P$; and consequently $\mathcal{R}_{\mathrm{GKP}}$ does not exist either. In particular, although the topology proposed by Rácz [16] yields the intuitively expected boundary structure for Taub's spacetime, it also has this defect.

4. Naked TIP

If an explicit identification rule were given, then it would be easier to construct the causal boundary for a given spacetime. Moreover, beyond this practical advantage, the boundary structure would be more transparent. Thus, following Budic and Sachs [4], we would like an identification rule $\mathcal{R}$ given explicitly as far as possible.

In this section a construction is presented that might be an appropriate candidate for such an identification. As is suggested by examples (e.g. the Reissner–Nordström, Kerr, anti-de Sitter spacetimes), certain preboundary points on the naked part of ∂^+ should be identified with certain preboundary points of the naked part of ∂^-. A TIP P is said to be naked if, for some point p of M, $P \subset I^-(p)$ [17, 18]. One can show that ∂^+ does not have a naked element iff M is globally hyperbolic [19] and this is also equivalent to $\uparrow P = \varnothing$ for every TIP $P \in \partial^+$. Since global hyperbolicity is time symmetric, the existence of a naked TIP implies the existence of a naked TIF. The next proposition shows that slightly more is true.

Proposition 4.1. If $P \in \partial^+$ is naked then there is a naked TIF F such that for $\forall q \in F$ $P \subset I^-(q)$ (such a TIF is called a naked counterpart of P) and F is maximal, i.e. F is not a proper subset of any naked counterpart of P.

Proof. Let $\gamma: [0, 1) \to M$ be a timelike generator of P and $\{t_n\}$ a sequence in $[0, 1)$ with no accumulation point and $\lim_{n\to\infty} t_n = 1$. P is naked and thus there is a point $p \in M$ such that, for all $n \in \mathbb{N}$, $\gamma(t_n) \ll p$. Let λ_n be a timelike curve from $\gamma(t_n)$ to p. γ is future endless and thus there is a past endless non-spacelike limit curve λ of the family $\{\lambda_n\}$ through p. M is future distinguishing and therefore $F' := I^+[\lambda]$ is a TIF such that for $\forall q \in F'$, $P \subset I^-(q)$. To prove maximality, we use Zorn's lemma (see also [20]). Let $\mathcal{F}$ be the collection of the naked counterparts of P. $\mathcal{F}$ is not empty and is a partially ordered set with respect to the inclusion relation. If $\mathcal{F}_0$ is a linearly ordered subset of $\mathcal{F}$ then $F_0 := \cup\, \mathcal{F}_0$ is a TIF and a naked counterpart of P. Thus $F_0 \in \mathcal{F}$ and hence, because of Zorn's lemma, $\mathcal{F}$ has a maximal element, i.e. each naked TIP has a maximal naked counterpart.

In general there may be naked TIP with more than one maximal naked counterparts. Based on the definitions one can prove easily the next statement.

Proposition 4.2. For a naked TIP P $\uparrow P = \bigcup_\alpha F_\alpha$, where the F_α are the maximal naked counterparts of P.

Note that if F is a maximal naked counterpart of P, then P is not necessarily a maximal naked counterpart of F.

Now, we turn to the problem of finding an appropriate identification. Of course, a relation is needed by means of which one can identify a naked TIP with those naked TIF only, which are naked counterparts of the TIP. Trivially, we wish to identify any naked TIP with some of its naked counterparts only if 'they lie arbitrarily close to each other', i.e. they are maximal with respect to each other.

Following the idea above we define the relation $\sim$: for the terminal indecomposable past set P and future set F, we write $P \sim F$ if they are maximal naked counterparts of each other. A naked TIP need not to be $\sim$-related to any naked TIF or may be $\sim$-related to more than one naked TIF. However, one can show easily that each naked TIP is contained in a naked TIP being $\sim$-related to some naked TIF: if F is a naked counterpart of the naked TIP P then there is a naked TIP P_0 and a naked TIF F_0 such that $P \subset P_0$, $F \subset F_0$ and $P_0 \sim F_0$. If P is a TIP and F is a TIF such that $P = \downarrow F$ and $F = \uparrow P$ then, as a corollary to proposition 4.2, $P \sim F$, i.e. our relation $\sim$ is an extension of the identification proposed by Budic and Sachs [4] for causally continuous spacetimes.

Until this point of the present section only the past and future distinguishing conditions have been used, without the strong causality condition $\sim$ yields unsatisfactory results. Let the strong causality condition be violated at $p \in M$ and define $P := I^-(p)$ and $F := I^+(p)$. If we cut out p from M then F becomes a naked TIF and P becomes a naked TIP in $M' := M - \{p\}$. While we think of F and P as representing the same point of the boundary of M', $P \not\sim F$. If the strong causality condition holds, then such pathologies cannot occur.

We define $\mathscr{R}$ as the smallest equivalence relation generated by $\sim$: for $\forall X \in M^{\#}$, let $(X, X) \in \mathscr{R}$ and for $B, B' \in \partial^+ \cup \partial^-$, $B \neq B'$, let $(B, B') \in \mathscr{R}$ if for a finite number of preboundary points $B_1, \ldots, B_r$, $B \sim B_1 \sim \ldots \sim B_r \sim B'$ holds.

In Taub's spacetime [3] our equivalence does not shrink the one-parameter family of pairs of null-finite TIP and TIF into a single point; it identifies only the TIP and TIF labelled by the same parameter value c, according to our intuitive picture. Recalling that Taub's spacetime is causally continuous, and hence is stable causal too [14], every non-spacelike curve has a unique past and future endpoint in $(M^{\#}/\mathscr{R}, \mathscr{T}^{\#}_{\mathrm{GKP}}/\mathscr{R})$ (proposition 2.3(b)). Figures 3 and 4 show two-dimensional stable causal spacetimes, which are additional examples for the difference of $\mathscr{R}$ and $\mathscr{R}_{\mathrm{GKP}}$. The spacetime of figure 3 is obtained from the half Minkowski plane by cutting out two series of closed segments $L_1, L_2, \ldots$, and $L'_1, L'_2, \ldots$. Here F and P are identified by our $\mathscr{R}$, but not by $\mathscr{R}_{\mathrm{GKP}}$, as they are T_2 separated in $\mathscr{T}^{\#}_{\mathrm{GKP}}/\mathscr{R}_{\mathrm{GKP}}$. In figure 4, which is the spacetime given by figure 9 of [2], the situation is just the reverse of the previous one: $\mathscr{R}_{\mathrm{GKP}}$ identifies F, P, F' and P', while our $\mathscr{R}$ glues together only F with P and F' with P'.

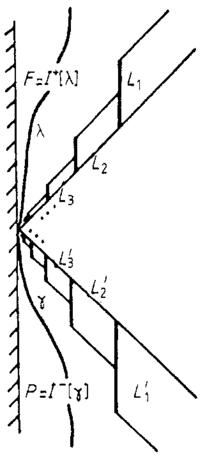

Figure 3. The TIP P and the TIF F are identified by $\mathscr{R}$ but not by $\mathscr{R}_{\mathrm{GKP}}$.

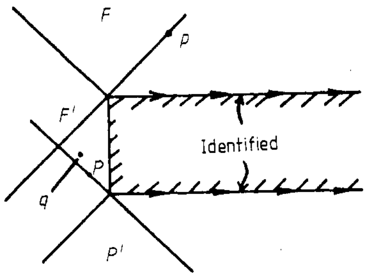

Figure 4. The TIP P, P' and the TIF F, F' are all identified by $\mathcal{R}_{\text{GKP}}$, but our $\mathcal{R}$ identifies F only with P and F' only with P'.

Of course, the topology $\mathcal{T}^{\#}_{\text{GKP}}/\mathcal{R}$ is not T_2, but the non-spacelike curves have unique endpoints.

5. PIP are naked

To identify PIP with the corresponding PIF the relation $\mathcal{R}_0$, and to identify points of $\partial^+ \cup \partial^-$ the relations $\sim$ and $\mathcal{R}$ were introduced. The completed space $\bar{M}$, therefore, was obtained from $M^+ \cup M^-$ in two steps. Thus it would be simpler and more convincing if the 'product' of equivalences $\mathcal{R}$, $\mathcal{R}_0$ could be defined in a unified single procedure.

In the present section we show that the notion of naked counterpart, as well as the relations $\sim$, $\mathcal{R}$ themselves can be extended to the IP and IF, and for PIP and PIF of a strongly causal spacetime the extended relations $\sim$, $\mathcal{R}$ are just the equivalence $\mathcal{R}_0$, i.e. both $\sim$, $\mathcal{R}$ are extensions of $\mathcal{R}_0$.

For any point $p \in M$ and the PIP $P = I^-(p)$ there is a point $q \in M$ such that $P \subset I^-(q)$, i.e. PIP are naked. Naked counterparts and maximal naked counterparts of a naked IP can be defined too. For naked IP and IF, $\sim$ will be defined in a similar way as we have done for TIP and TIF. While a PIP may have a maximal naked counterpart which is a TIF, no naked TIP can have any maximal naked counterpart being a PIF. (The proof is a simple application of the well known 'limit-curve technique' [6].) Consequently, no inner point can be $\sim$-related to any preboundary point, and hence the relation $\sim$ defined for IP and IF is an extension of that introduced in § 4 for TIP and TIF. To prove that this $\sim$ is an extension of $\mathcal{R}_0$ too, the strong causality condition is needed, as is suggested by figure 38 of [6].

Proposition 5.1. (a) M is strongly causal iff $I^+(p) \sim I^-(p)$ for $\forall p \in M$.
(b) If M is strongly causal then $I^-(p) \sim F$ iff $F = I^+(p)$.

Proof. (a) If the strong causality condition is violated at $p \in M$ then there is a neighbourhood U of p such that for every neighbourhood V_n of p contained in U there are points $x_n \in I^-(p, V_n)$, $y_n \in I^+(p, V_n)$ and a timelike curve λ_n from x_n, leaving U and returning to y_n [5, 6]. The family $\{\lambda_n\}$ has a non-spacelike limit curve λ through p, which is actually a null geodesic. Trivially, $I^+(p)$ is a naked counterpart of $I^-(p)$ and, using the family $\{\lambda_n\}$, one can show that $I^+(r)$ is a naked counterpart of $I^-(p)$ too for $\forall r \in J^-(p) \cap (\lambda - \{p\})$. Now $I^+(p) \subset I^+(r)$. But then $I^+(p)$ can be a maximal naked counterpart of $I^-(p)$ only if $I^+(p) = I^+(r)$ which, however, contradicts the future distinguishing condition.

Conversely, suppose, on the contrary, that for some point p and future directed timelike curve σ (with or without endpoint) $F = I^+[\sigma]$ is a naked counterpart of $I^-(p)$ and $I^+(p) \subset F \not\subset I^+(p)$. Then there is a point $r \in \sigma \cap F$ and a causally convex neighbourhood V of p such that $(I^-(r) \cup F) \cap V = \emptyset$. If $y \in I^+(p) \cap V$ then there is a point $s \in \sigma \cap I^-(r) \cap F$ such that $s \ll y$, i.e. there is a timelike curve μ from s to y. On the other hand, however, F is a naked counterpart of $I^-(p)$. Thus for $\forall x \in I^-(p) \cap V x \ll s$, i.e. there is a timelike curve meeting V twice, which contradicts the causal convexity of V.

(*b*) If $F = I^+(p)$ then $I^-(p) \sim F$, according to the first part of this proposition.

Conversely, suppose that $I^-(p) \sim F$, where F is a naked IF. Then F must be a PIF, i.e. $F = I^+(q)$ for some $q \in M$. Since $I^+(p) \sim I^-(p)$, one has to show that $p = q$, i.e. the uniqueness of $I^+(p)$. On the contrary, suppose $p \neq q$. If $\{p_n\}$ and $\{q_m\}$ are sequences of points from $I^-(p)$ and $I^+(q)$, converging to p and q, respectively, then there is a timelike curve λ_{nm} from p_n to q_m. The family $\{\lambda_{nm}\}$ has a future directed non-spacelike limit curve λ from p. Since both $I^+(q)$ and $I^+(p)$ are maximal naked counterparts of $I^-(p)$, $I^+(q) \not\subset I^+(p) \not\subset I^+(q)$, implying that λ is a null geodesic. If $r \in \lambda - \{p\}$, then $I^-(p) \subset I^-(r)$ and, since r is a limit point of $\{\lambda_{nm}\}$, $I^-(r)$ is a naked counterpart of $I^+(q)$. Consequently, $I^-(p)$ can be a maximal naked counterpart of $I^+(q)$ only if $I^-(r) = I^-(p)$ which, however, contradicts the past distinguishing condition.

6. Causal structure for $\bar{M}$

With the natural causal relations $\ll$, $<$ $(M, \ll, <)$ is a causal space. Furthermore $M^\pm$ can also be equipped with causal space structure such that the mappings $I^\pm$ are causal morphisms [2, 6, 8].

As far as possible, our aim is to extend the causal relations $\ll$, $<$ to the completed space $\bar{M}$. Although the causality relation between two points of the spacetime can change if boundary points are added to M, the chronology relation, however, is expected to remain the same, because the chronological futures and pasts are open in M and $i(M)$ is dense in $\bar{M}$. This implies that, from our 'extension' point of view, neither of the definitions of [2, 4] for these relations is acceptable: the points q, p in figure 4 are not causally related in M, but $I^+(q)$, as a PIP, chronologically predicts $I^+(p)$ if the chronology coming from [2, 4] is used.

Recalling that each point m of $\bar{M}$ is a class $[P_1, P_2, \ldots; F_1, F_2, \ldots]$ of $\mathcal{R}$-equivalent IP and IF, define $m \overline{\ll} m'$ if, for some $F_\alpha \in \pi^{-1}(m)$ and $P'_\mu \in \pi^{-1}(m')$, $F_\alpha \cap P'_\mu \neq \emptyset$ (see also [10]). Based on these definitions one can easily prove the next statement.

Proposition 6.1. (*a*) $\overline{\ll}$ is an antireflexive and transitive relation on $\bar{M}$.

(*b*) For $p, q \in M$ $p \ll q$ iff $i(p) \overline{\ll} i(q)$.

Thus $\bar{M}$ with $\overline{\ll}$ is a chronological space [21] into which M is chronologically embedded by i. For example, since our $\mathcal{R}$ does not identify the boundary points $b := \pi(P) = \pi(F)$ and $b' := \pi(P') = \pi(F')$ in figure 4, one has $b' \overline{\ll} i(q) \overline{\ll} b$; while if $\mathcal{R}_{\mathrm{GKP}}$ were used then the structure of $M^\#/\mathcal{R}_{\mathrm{GKP}}$ would contradict to any reasonable chronology relation [2].

Using the chronology $\overline{\ll}$ one can define causality $\precsim^B$ on $\bar{M}$ [8]: $m \precsim^B m'$ if $I^+(m', \bar{M}) \subset I^+(m, \bar{M})$ and $I^-(m, \bar{M}) \subset I^-(m', \bar{M})$, where $I^+(m, \bar{M}) := \{m' \in \bar{M} \mid m \overline{\ll} m'\}$. Then $(\bar{M}, \overline{\ll}, \precsim^B)$ is a causal space for which i is a causal morphism and is a causal isomorphism iff M is causally simple. However, in certain situations we would like to consider the points m, m' as causally related even if either or both of the defining relations of $\precsim^B$ do not hold, or sometimes as causally unrelated even if both of the defining relations hold. Thus we do not consider $\precsim^B$ as a satisfactory causality relation on $\bar{M}$.

7. Conformally embedded spacetimes

If the spacetime can be embedded, in some sense, into a larger spacetime then, as far as possible, causal boundary construction is expected to reproduce the boundary structure obtained from the embedding [6, 9, 22]. In this section some remarks are given about how the identification $\mathcal{R}$ works if the spacetime is conformally embeddable.

The spacetime (M, g) is said to be (totally) conformally embedded into the strongly causal spacetime $(\tilde{M}, \tilde{g})$, if there is an embedding $\theta: M \to \tilde{M}$ and a function $\Omega: \tilde{M} \to [0, \infty)$ such that $\theta^* \tilde{g} = (\theta^* \Omega^2) g$; furthermore, for $\forall P \in \partial^+$ there is a point $\tilde{p} \in \tilde{M}$ such that $\theta(P) = I^-(\tilde{p}, \theta(M))$, and similarly for $\forall F \in \partial^- \exists \tilde{q} \in \tilde{M}$ such that $\theta(F) = I^+(\tilde{q}, \theta(M))$. Of course, $\tilde{p}, \tilde{q} \in \partial\theta(M)$ (the boundary of $\theta(M)$ in the manifold topology of $\tilde{M}$). Based on arguments similar to those used in proposition 5.1, one can prove that $P \sim F$ implies $\tilde{p} = \tilde{q}$, and conversely, if $\tilde{p} = \tilde{q}$ but $P \not\sim F$, then P and F are not naked counterparts of each other either. These results suggest that, at least in the present case, $\sim$ is the minimal identification we have to carry out, but it does not seem to be reasonable to identify further points.

8. Conclusions

To assign causal boundary ∂_c to the spacetime M, i.e. to obtain the (causally) completed spacetime $\bar{M}$, a topology $\mathcal{T}^\#$ and an identification rule $\mathcal{R}$ is needed on an auxiliary set $M^\#$. For certain properties of the topology $\bar{\mathcal{T}}$ of $\bar{M}$, e.g. separation properties, necessary and sufficient conditions can be given in terms of $\mathcal{T}^\#$ and $\mathcal{R}$.

A detailed examination of the topology $\mathcal{T}^\#_{\text{GKP}}$, a possible candidate for $\mathcal{T}^\#$, shows that in general there might be points of M which are not unique endpoints, in the topology $\mathcal{T}^\#_{\text{GKP}}/\mathcal{R}$, of the timelike curves ending there, for any $\mathcal{R}$. This implies, first, that the identification $\mathcal{R}_{\text{GKP}}$ might not exist in general, and hence a new identification rule is needed; second, the problem cannot be solved if the topology $\mathcal{T}^\#_{\text{GKP}}$ is used further on and only the equivalence $\mathcal{R}$ is changed.

There might be, however, a more annoying consequence of this difficulty, namely that it does not seem to be possible to find an appropriate causal topology $\mathcal{T}^\#$ if 'the strong causality condition is violated at a preboundary point', i.e. when the neighbourhoods of a preboundary point B are 'not concentrated on B'. (Think of the Alexandrov neighbourhoods of a strong causality violating point.) Though such situations cannot occur for stable causal spacetimes but, in general, we might have to give up our claim of unique endpoints (and hence the Hausdorffness too), or we have to restrict ourselves to stable causal spacetimes.

To overcome half of the difficulties we propose a new explicit identification rule for general strongly causal spacetimes. Our $\mathcal{R}$, which is an extension of the equivalence gluing together PIP with the corresponding PIF, yields the intuitively expected point set structure of ∂_c onto which the chronology relation can be extended. The equivalence $\mathcal{R}$ is built up from elementary TIP–TIF gluings. Thus at this point one can ask the rather speculative question whether $\mathscr{R}$ is complete or, for example, certain TIP–TIP and TIF–TIF gluings should be included as well. There is some indication, suggested by examples like that shown by figure 2, that $\mathscr{R}$ may be incomplete, but it is not clear how it ought to be completed. However, any reasonable identification should contain our $\mathscr{R}$.

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