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Causal boundary for strongly causal spacetimes: II

L B Szabados

Central Research Institute for Physics, PO Box 49, H-1525 Budapest 114, Hungary

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Abstract. In a previous paper an analysis of the general structure of the causal boundary constructions and a new explicit identification rule, built up from elementary TIP–TIF gluings, were presented. In the present paper we complete our identification by incorporating TIP–TIP and TIF–TIF gluings as well. An asymptotic causality condition is found which, for physically important cases, ensures the uniqueness of the endpoints of the non-spacelike curves in the completed spacetime.

1. Introduction

To study some special problems in general relativity the introduction of certain kinds of boundary to spacetime seems to be useful. Several boundary constructions have been made [1–5] but a large class of them has unacceptable topological features [6]. The constructions based on causality [3–5], however, are free of these kinds of defects. Moreover, causal boundary constructions are probably the simplest and most transparent ones.

Recently, however, certain difficulties with the traditional construction [4] have been pointed out. If the spacetime is not stable causal, then the traditional construction does not always exist [7]. Furthermore, it does not always yield the intuitively expected boundary structure even if the spacetime is causally continuous. Although the collection of null finite points of the boundary of Taub's plane-symmetric static spacetime is expected to be a one-dimensional set, it consists of a single point only [8]. On the other hand, there are different boundary points on the causal boundary, obtained in the traditional way, of the four-dimensional half Minkowski spacetime that we would like to consider as a single boundary point [9].

In spite of these difficulties we do not think that the causal boundary construction is completely unsatisfactory. The construction is made in two steps. First, by means of past and future endless non-spacelike curves, additional 'ideal' or 'preboundary' points are added to the spacetime, and then certain preboundary points are identified. The difficulties above are due only to the very implicit nature of the traditional identification rule [10]; a new explicit identification is therefore needed.

In the previous paper [7] the general framework for the causal boundary constructions and the topology of Geroch *et al* [4] were studied and a new explicit identification rule was proposed. That identification is built up from elementary gluings identifying certain TIP with certain TIF of the naked part [11] of the preboundary. However, the problem has not been solved completely; two open questions remain. First, as examples suggest, further preboundary points should be identified, namely certain TIP–TIP and

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TIF–TIF gluings should also be included [10]. Thus one has to clarify how the new gluings ought to be defined. Second, we did not have any statement on the separation properties of the resulting completed spacetime. In fact, without a certain 'asymptotic causality condition' and additional TIP–TIP and TIF–TIF gluings, one cannot hope to obtain nice topological properties for the completed spacetime. On physical grounds, it seems reasonable to require the uniqueness of the endpoints of the world lines [7]. Hence one should determine what additional 'asymptotic causality condition' is able to guarantee the unique endpoints of the non-spacelike curves.

The present paper is the second part of our previous work [7], and now we try to answer the open questions above. To recall the statements we need and to fix the notations, in the second section we review the main points of the construction, and we introduce the causal separation axiom T_c. The third section is devoted to the analysis of certain asymptotic causal pathologies of the spacetime. To handle these pathologies we extend the definition of the strong causality condition to the preboundary of M, and show that the violation of this condition yields non-T_c-separated points. If the extended strong causality condition holds then, in terms of the identification rule, we prove necessary and sufficient conditions for the completed spacetime $(\bar{M}, \bar{\mathcal{T}})$ being a T_c space. Section 4 is a brief review of the TIP–TIF gluing we introduced in our earlier paper [7]. In § 5 the new elementary gluings, identifying TIP with TIP and TIF with TIF, will be introduced and a number of their causal and topological properties will be described. In § 6 we build up our identification rule $\bar{\mathcal{R}}$ and show that the completed spacetime is a T_c space if the extended strong causality condition holds and each $\mathcal{R}$-equivalence class is finite. Finally, certain concepts of the construction will be discussed.

Throughout this paper the standard matter of the global technique [12, 13] will be used; thus we omit the continuous references to the well known statements. The standard reference for the theory of causal boundary is [4] (see also [7]). Our conventions and notations are the same as those used in [7, 12] unless otherwise stated. Spacetime is assumed to be time oriented and strongly causal, and the past and future sets are assumed to be open [13].

2. Preliminaries and the causal separation axiom

Let M denote the spacetime manifold and M^+ and M^- the collections of IF and IP, respectively. If ∂^+ and ∂^- denote the collection of TIP and the collection of TIF, respectively, and $i^\pm$: $M \to M^\pm : p \mapsto I^\mp(p)$, then $M^\pm$ are the disjoint unions $i^\pm(M) \cup \partial^\mp$. The past and future distinguishing conditions hold on M iff $i^\pm$ are injective maps. Then the points of M can be represented by means of their chronological futures and/or pasts; and the points of ∂^+ and ∂^- are interpreted as additional points at the future and past boundary of M, respectively. To have the past and future boundary at the same time, the disjoint union $M^+ \cup M^-$ should be considered. But $i^+(p)$ and $i^-(p)$ represent the same point of M; thus they must be identified. The resulting set $M^{\#}$ is the disjoint union $i(M) \cup \partial^+ \cup \partial^-$, where $i(p)$ stands for the identified pair $(i^+(p), i^-(p))$ and the map $i: M \to M^{\#}$ is injective [4].

$\partial^+ \cup \partial^-$ does not have the expected point-set structure yet: further identifications have to be carried out [4, 10]. The mathematical form of such an identification is an equivalence relation $\mathcal{R}$ on $M^{\#}$ that contains the identity relation and is diagonal on

$i(M)$. If $\pi: M^{\#} \to M^{\#}/\mathcal{R}$ is the canonical projection associated with $\mathcal{R}$, then the completed spacetime $\bar{M}$ is expected to be obtained as a quotient $M^{\#}/\bar{\mathcal{R}}$ for some $\bar{\mathcal{R}}$, and $\partial_c := M^{\#}/\bar{\mathcal{R}} - \pi \circ i(M)$ is interpreted as the causal boundary of M.

The causal boundary construction is expected to yield not only the right point-set structure of the boundary, but the topological properties as well. A natural causal topology, namely the Alexandrov topology $\mathcal{T}$ is given on M. To describe the topological properties of ∂_c, a topology $\bar{\mathcal{T}}$ is needed on $\bar{M}$. $\bar{\mathcal{T}}$ can always be considered as the quotient topology $\mathcal{T}^{\#}/\bar{\mathcal{R}}$ of an appropriate $\mathcal{T}^{\#}$ on $M^{\#}$. Although there is a great number of candidates for $\mathcal{T}^{\#}$, which are defined by means of causality, throughout this paper $\mathcal{T}^{\#} = \mathcal{T}^{\#}_{\mathrm{GKP}}$, i.e. the topology of Geroch *et al* [4], will be used. One can show [7] that $\pi \circ i : (M, \mathcal{T}) \to (M^{\#}/\mathcal{R}, \mathcal{T}^{\#}/\mathcal{R})$ is an open dense embedding for any equivalence relation $\mathcal{R}$. Furthermore, for any non-spacelike curve γ in M, $\pi(I^-[\gamma])$ and $\pi(I^+[\gamma])$ are future and past endpoints of $\pi \circ i \circ \gamma$ in $M^{\#}/\mathcal{R}$, respectively. Before considering the separation properties of $M^{\#}/\mathcal{R}$, we have to clarify which separation axiom can be expected to hold.

The weakest separation axiom that should be expected is T_1, but, as for example the difficulties with Taub's plane-symmetric static-vacuum solution suggest, the requirement of the Hausdorffness of $\bar{M}$ is too strong. The requirement of the uniqueness of the future and past endpoints of the non-spacelike curves, however, seems to be a reasonable 'physical' separation axiom. The points $x, y \in M^{\#}/\mathcal{R}$ are T_c separated in $\mathcal{T}^{\#}/\mathcal{R}$ if no irreducible representative X of $\pi^{-1}(x)$ and no irreducible representative Y of $\pi^{-1}(y)$ has a non-spacelike generator γ such that both x and y would be either future or past endpoints of $\pi \circ i \circ \gamma$ in $\mathcal{T}^{\#}/\mathcal{R}$. The space $(M^{\#}/\mathcal{R}, \mathcal{T}^{\#}/\mathcal{R})$ is said to be a T_c space if every two different points of $M^{\#}/\mathcal{R}$ are T_c separated in $\mathcal{T}^{\#}/\mathcal{R}$; and this is equivalent to the uniqueness of the future and past endpoints of the curves $\pi \circ i \circ \gamma$ in $\mathcal{T}^{\#}/\mathcal{R}$ for any non-spacelike curve γ in M. One can show that axiom T_c is weaker than T_2 but is stronger than T_1, i.e. axiom T_2 implies T_c and axiom T_c implies T_1. (For the motivations see [1, 14–17], and especially [18].)

Assuming that the strong causality condition holds on M, one can show [7] the following separation properties of $(M^{\#}, \mathcal{T}^{\#})$.

(i) Any two different inner points are T_2 separated.

(ii) Let $P \in \partial^+$ and $q \in M$. If $q \notin \partial{\uparrow}P = \overline{{\uparrow}P} - {\uparrow}P$, then P and $i(q)$ are T_2 separated, and if $q \in \partial{\uparrow}P$ then P and $i(q)$ are T_1 separated.

(iii) If $P \in \partial^+$, $F \in \partial^-$ and $F \not\subset {\uparrow}P$ then F and P are T_2 separated, and if $F \subset {\uparrow}P$ then P and F are T_c separated.

(iv) Let $P', P \in \partial^+$. If $P' \not\subset P \not\subset P'$ then P' and P are T_c separated. If $P' \subset I^-(p)$ for some $p \in P$, then P' and P are T_2 separated. If $P' \subset P$ and $P' \subset I^-(p)$ for no point p of P then P' and P are T_0 'separated', or more precisely, P cannot be a future endpoint of any non-spacelike curve generating P'.

However, the separation properties of $(M^{\#}/\mathcal{R}, \mathcal{T}^{\#}/\mathcal{R})$ depend on the global causal pathologies of M and the equivalence $\mathcal{R}$ too.

3. The asymptotic causality condition

To study the separation properties of the quotient spaces, it seems useful to consider first the separation of inner and boundary points and then the separation of boundary points from each other. (If the strong causality condition holds on M, which will be assumed later, the inner points are always T_2 separated from each other.)

One can show [7] that, for $\forall q \in M$, $B \in \partial^+ \cup \partial^-$ and any $\mathcal{R}$, $\pi(B)$ and $i(q)$ are T_1, but not necessarily T_c separated in $\mathcal{T}^{\#}/\mathcal{R}$. (Since $\mathcal{R}$ is diagonal and hence π is injective on $i(M)$, we omit π; i.e. we write $i(q)$ and $i \circ \gamma, \ldots$, instead of $\pi \circ i(q)$ and $\pi \circ i \circ \gamma, \ldots$, respectively.) Figure 1 shows a two-dimensional spacetime $\tilde{M}$, obtained from M by the extension E, where the boundary point $\pi(P)$ is not T_c separated from $i(q)$, $q \in \partial \uparrow P$, for any $\mathcal{R}$. Their T_c separation depends on the existence of certain neighbourhoods of the points of M. One can show that the following statements are equivalent (see [7]).

(i) b and $i(q)$ are T_2 separated in $\mathcal{T}^{\#}/\mathcal{R}$ for all $\mathcal{R}$ and $b \in \partial^+ \cup \partial^-/\mathcal{R}$, $q \in M$.

(ii) $(M^{\#}/\mathcal{R}_m, \mathcal{T}^{\#}/\mathcal{R}_m)$ is a T_2 space, where $\mathcal{R}_m$ is the maximal identification yielding a single point set for the boundary.

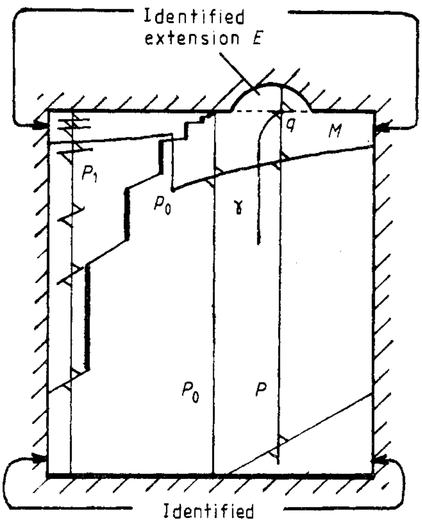

Figure 1. M is a two-dimensional cylinder from which a countable infinity of vertical closed segments is omitted and the causal structure is shown by the light cones. If the extension E is not carried out then (M, g) is stable causal and the $\mathcal{T}^{\#}$-open neighbourhoods of P_0 have the form $(I^+(p_0))^{\text{int}}$, $p_0 \in P_0$. Both P and P_0 are endpoints of $i \circ \gamma$ in $\mathcal{T}^{\#}$.

(iii) Each point q of M has a $\mathcal{T}$-open neighbourhood V such that there is a $\mathcal{T}^{\#}$-open neighbourhood U_B of every $B \in \partial^+ \cup \partial^-$ for which $U_B \cap i(V) = \varnothing$. (This V is called a universal neighbourhood of q.)

The existence of universal neighbourhoods implies the strong causality condition, and is implied by the stable causality condition on M. Hence the existence of the universal neighbourhoods implies the T_c separation of inner and boundary points, but it does not rule out the 'bad' separation of the boundary points from each other.

The separation properties of $(\bar{M}, \bar{\mathcal{T}})$ depend also on the equivalence $\bar{\mathcal{R}}$ we use. For example, if $\bar{\mathcal{R}}$ did not identify the TIP P and P' in the two-dimensional spacetime shown by figure 2, then $(\bar{M}, \bar{\mathcal{T}})$ would not be a T_c space. In fact [7], if each point of M has a universal neighbourhood then $(M^{\#}/\mathcal{R}, \mathcal{T}^{\#}/\mathcal{R})$ is a T_c space iff the non-T_c-separated points of $(M^{\#}, \mathcal{T}^{\#})$ are $\mathcal{R}$ equivalent and the equivalence classes $[X] := \{Y \in M^{\#} | (X, Y) \in \mathcal{R}\}$, $X \in M^{\#}$, are all closed in $(M^{\#}, \mathcal{T}^{\#})$. Thus the simplest way to obtain a T_c space for $\bar{M}$ would be to define $\bar{\mathcal{R}}$ as the smallest closed equivalence

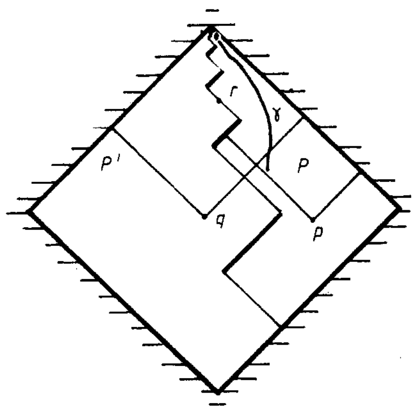

Figure 2. This stable causal spacetime is a part of the quarter Minkowski plane from which a countable infinity of closed segments is removed. Here P and P' are endpoints of $i \circ \gamma$ in $\mathscr{T}^{\#}$.

relation identifying the non-T_c-separated preboundary points. Nevertheless, this $\bar{\mathscr{R}}$ would also be implicitly defined and would not yield the expected boundary structure. Moreover, $\bar{\mathscr{R}}$ would not necessarily exist if the points of M did not have universal neighbourhoods.

But what criteria for the boundary structure can be adopted? If the spacetime can be conformally extended into a strongly causal spacetime, then, as far as possible, we want a boundary structure for ∂_c like that obtained from the extension [7, 19]. (For a discussion of this concept see § 7.) As a special case, we would like to recover the boundary of the well known special exact solutions obtained by the conformal technique [11, 12, 14]. Thus, for example, if the extension E is not carried out, then $\bar{\mathscr{R}}$ must not identify all the TIP at the 'top edge' of the two-dimensional spacetime M shown by figure 1. Only P_0 and P_1 are allowed to be identified. (Of course, the 'top edge' of M would be shrunk if the infinity of the closed segments were not cut, since otherwise the whole 'top edge' would be a single TIP. But then the locally extended spacetime $\tilde{M}$ would not be past-distinguishing.) Although M is stable causal, the TIP P_0 and P are not T_c separated in $\mathscr{T}^{\#}$. This topological defect is due to the 'violation of the strong causality condition at the preboundary point P_0' [7]. Consequently, one cannot hope for nice separation properties for $\tilde{M}$ without a certain asymptotic causality condition excluding such neighbourhoods of the preboundary points B that are 'not concentrated on B'.

We will say that the extended strong causality condition is violated at the IP $P_0 \in M^-$, if there exists an IP P such that $P_0 \subset P \not\subset P_0$ and for $\forall p \in P$ there is a set $S \subset M$ with $P_0 = I^-[S]$ and, for $\forall x \in I^+[S]$, there is a future-directed timelike curve starting at p, leaving P and ending at x. Recalling that the past-distinguishing condition is assumed to hold on M, we have the following lemma.

Lemma 3.1. The extended strong causality condition holds at the PIP $P_0 := I^-(r) \in M^-$ iff the (ordinary) strong causality condition holds at $r \in M$.

Proof. Let the strong causality condition hold at $r \in M$ and suppose, on the contrary, that the extended strong causality condition is violated at $I^-(r) \in M^-$, and let P be the additional IP specified in the definition above. Then r has a causally convex neighbourhood W such that, for some timelike generator γ of P, $\gamma\cap W=\emptyset$ holds. For $\forall y\in W\cap I^-(r)$ there is a point $p\in\gamma$ for which $y\ll p$, and for this p there is a set $S\subset M$ with $P_0=I^-[S]$. Due to the past-distinguishing condition, $r\in\bar{S}$, which implies that $W\cap I^+[S]\neq\emptyset$. But then, for $\forall x\in W\cap I^+[S]$, there is a timelike curve from p to x, implying the violation of the strong causality condition at $r\in M$.

If the strong causality condition is violated at $r\in M$, then there is a future inextendible null geodesic γ from p along which the strong causality condition is violated, and let $q\in J^+(r)\cap\gamma-\{r\}$ and $P:=I^-(q)$. Then $P_0\subset P$ and, because of the past-distinguishing condition, $P\not\subset P_0$. For $\forall p\in P$ let $S:=\{r\}$ and let x be any point of $I^+[S]$. Since the strong causality condition is violated at r and q is a limit point of a family of timelike curves meeting arbitrarily small neighbourhoods of r more than once, there is a future-directed timelike curve starting at p, leaving P and ending at x.

Since the ordinary strong causality condition is time symmetric, this lemma and its dual imply that the extended strong causality condition holds at the PIP $I^-(r)$ iff it holds at the PIF $I^+(r)$. Therefore the extended strong causality condition is well defined on $M^{\#}$ and, together with the past- and future-distinguishing conditions, this is just the ordinary strong causality condition on M and an additional asymptotic causality condition on $\partial^+\cup\partial^-$. As figure 1 shows, stable causality does not imply the asymptotic causality condition, but, as our next statement states, causal continuity does.

Proposition 3.2. If M is causally continuous then the extended strong causality condition holds on $M^{\#}$.

Proof. The ordinary strong causality condition follows from causal continuity [20]; thus, on the contrary, suppose that the extended strong causality condition is violated at the TIP P_0. If P is the IP specified by the definition, then, due to the ordinary strong causality condition, there exists a non-empty compact set $K:=\overline{I^+(q)\cap I^-(p)}$ in $P-\bar{P}_0$. For this $p\in P-\bar{P}_0$ there is a set $S\subset M$ such that $P_0=I^-[S]$ and, for $\forall x\in I^+[S]$, there is a timelike curve from p to x. Let $s\in S$ and U be any open neighbourhood of s. Then $K\subset M-\overline{I^-(s)}$ but, for any $x\in U\cap I^+(s)$, $K\subset I^-(x)$ would follow. Therefore i^- would not be outer continuous at s [20], i.e. M could not be causally continuous.

The violation of the asymptotic causality condition yields non-T_c-separated IP; but excluding this causal pathology one can hope to obtain nice separation properties for $\bar{M}$, as follows.

Proposition 3.3. If the extended strong causality condition is violated at $P_0\in\partial^+$ then there is an IP P such that $P_0\subset P\neq P_0$ and P_0 is a future endpoint of $i\circ\gamma$ in $(M^{\#},\mathcal{T}^{\#})$ for every timelike generator γ of P.

Proof. The extended strong causality condition is violated at $P_0\in\partial^+$. Thus for some IP P, $P_0\subset P\neq P_0$, and any point $p\in P$ there is a set $S_p\subset M$ such that $P_0=I^-[S_p]$ and, for $\forall x\in I^+[S_p]$, there is a timelike curve starting at p, leaving P and ending at x. Let γ be any timelike generator of P with parameter domain $[0,1)$ and $q\in P_0$. Then, for some $\varepsilon>0$, $\gamma([1-\varepsilon,1))\subset I^+(q)$; i.e. $i\circ\gamma$ enters and remains in $(I^+(q))^{\mathrm{int}}$, which is a $\mathcal{T}^{\#}$-open neighbourhood of P_0. If P_0 were not an endpoint of $i\circ\gamma$, then there would be an IF F such that $P_0\in F^{\mathrm{ext}}$ and $i\circ\gamma$ would leave F^{ext}; i.e. γ would enter $\bar{F}$. If $p\in\gamma\cap\bar{F}$ then $I^+(p)\subset F$, thus $F^{\mathrm{ext}}\subset(I^+(p))^{\mathrm{ext}}$, which would imply $P_0\in(I^+(p))^{\mathrm{ext}}$. But then, for $p\in\gamma\cap\bar{F}\subset P$, S_p could not exist with $P_0=I^-[S_p]$ and $I^+[S_p]\subset I^+(p)$.

Proposition 3.4. Let the extended strong causality condition hold on $M^{\#}$ and let γ be a non-spacelike curve.

(a) If $p \in M$ is the future endpoint of γ in $(M, \mathcal{T})$, then $i(p)$ is the unique future endpoint of $i \circ \gamma$ in $(M^{\#}/\mathcal{R}, \mathcal{T}^{\#}/\mathcal{R})$ for any $\mathcal{R}$.

(b) If γ is future endless in $(M, \mathcal{T})$ and $P := I^-[\gamma]$, then $\pi(P)$ is the unique future endpoint of $i \circ \gamma$ in $(M^{\#}/\mathcal{R}, \mathcal{T}^{\#}/\mathcal{R})$ iff $[P]$ is closed in $(M^{\#}, \mathcal{T}^{\#})$ and every future endpoint of $i \circ \gamma$ in $(M^{\#}, \mathcal{T}^{\#})$ is $\mathcal{R}$-equivalent to P.

Proof. Let $[0, 1)$ be the parameter domain of γ. First of all we show that the set $i \circ \gamma([0, \delta])$ is closed in $(M^{\#}, \mathcal{T}^{\#})$ for all $\delta \in (0, 1)$. Since $\gamma([0, \delta])$ is closed in $(M, \mathcal{T})$, the set $i(M - \gamma([0, \delta]))$ is a $\mathcal{T}^{\#}$-open neighbourhood of $i(z)$, $z \in M - \gamma([0, \delta])$, which does not intersect $i \circ \gamma([0, \delta])$. Now suppose, on the contrary, that there is a preboundary point, for example a TIP P'_0, such that every $\mathcal{T}^{\#}$-open neighbourhood of P'_0 intersects $i \circ \gamma([0, \delta])$. Let γ'_0 be a generator of P'_0 and $\{q'_n\}$ be a monotonic sequence on γ'_0 with no accumulation point in M. Then $I^+(q'_n) \cap \gamma([0, \delta]) \neq \varnothing$ and let $p'_n := \inf_{\ll_\gamma}\{w \in I^+(q'_n) \cap \gamma([0, \delta])\}$, where $\ll_\gamma$ is the natural ordering along the segment $\gamma([0, \delta])$. $\gamma([0, \delta])$ is compact in the manifold topology, thus $p'_n \in \gamma([0, \delta])$; moreover, there is an accumulation point p' of $\{p'_n\}$ in $\gamma([0, \delta])$. Then $p'_n \ll_\gamma p'$ for $\forall n \in N$, or there is a number $n_0 \in N$ such that $p'_n = p'$ for $\forall n > n_0$. If $p'_n \ll_\gamma p'$ holds for $\forall n \in N$, then $P'_0 \subset P' := I^-(p') \not\subset P'_0$. Moreover, for $\forall r' \in P'$, there is a set $S' \subset M$ with $P'_0 = I^-[S']$ and, for $\forall x \in I^+[S']$, there is a timelike curve starting at r', leaving P' and terminating at x; otherwise, for large enough n and for some $r' \in P'$, $(I^+(q'_n))^{\text{int}} \cap (I^+(r'))^{\text{ext}}$ would be a $\mathcal{T}^{\#}$-open neighbourhood of P'_0 which would not intersect $i \circ \gamma([0, \delta])$. This, however, contradicts the extended strong causality condition. If $p'_n = p'$ for any $n > n_0$, then γ enters $\uparrow P'_0$ at p' and, because of $P'_0 \in (I^+(p'))^{\text{ext}}$, $V := (I^+(q'_n))^{\text{int}} \cap (I^+(p'))^{\text{ext}}$ is a $\mathcal{T}^{\#}$-open neighbourhood of P'_0. But V does not intersect $i \circ \gamma([0, \delta])$, which is a contradiction again. Therefore $i \circ \gamma([0, \delta])$ is closed in $(M^{\#}, \mathcal{T}^{\#})$.

Let $q \in M$. If γ is any non-spacelike curve (with or without endpoint in M) for which q is not an endpoint in M, then $i(q)$ is not an endpoint of $i \circ \gamma$ in $(M^{\#}/\mathcal{R}, \mathcal{T}^{\#}/\mathcal{R})$ either. Thus it remains to show that no point of $\partial^+ \cup \partial^-/\mathcal{R}$ and $\partial^+ \cup \partial^-/\mathcal{R} - \{\pi(P)\}$ can be an endpoint of $i \circ \gamma$ in $(M^{\#}/\mathcal{R}, \mathcal{T}^{\#}/\mathcal{R})$ in the cases (a) and (b), respectively.

(a) Let p be the future endpoint of γ. Then $i(p)$ is a future endpoint of $i \circ \gamma$ in $(M^{\#}/\mathcal{R}, \mathcal{T}^{\#}/\mathcal{R})$. First suppose, on the contrary, that there is a preboundary point, for example a TIP P_0, which is an endpoint of $i \circ \gamma$ in $(M^{\#}, \mathcal{T}^{\#})$. Then for any $\mathcal{T}^{\#}$-open neighbourhood $V := (I^+(q))^{\text{int}} \cap F_1^{\text{ext}} \cap \ldots \cap F_l^{\text{ext}}$ of P_0, there is a parameter value $v \in [0, 1)$ for which $\gamma([v, 1)) \subset I^+(q) \cap (M - \overline{F_1}) \cap \ldots \cap (M - \overline{F_l})$, $q \in P_0$. This implies $P_0 \subset P := I^-(p) \not\subset P_0$; moreover, for any $r \in I^-(p)$, there is a set $S \subset M$ such that $P_0 = I^-[S]$ and, for any $x \in I^+[S]$, there is a timelike curve starting at r, leaving P and ending at x. This, however, contradicts the extended strong causality condition. Thus $i(p)$ is the unique future endpoint of $i \circ \gamma$ in $(M^{\#}, \mathcal{T}^{\#})$ and hence $\{i(p)\}$ is closed in $(M^{\#}, \mathcal{T}^{\#})$.

If $B \in \partial^+ \cup \partial^-$ then there is a $\mathcal{T}^{\#}$-open neighbourhood $\tilde{U}_B$ of B and a number $\delta \in (0, 1)$ such that $\tilde{U}_B \cap i \circ \gamma([\delta, 1)) = \varnothing$, because B is not an endpoint of $i \circ \gamma$ and $\mathcal{T}^{\#}$ is a causal topology [7]. If $U_B := \tilde{U}_B - i \circ \gamma([0, \delta]) - \{i(p)\}$ and $U := \bigcup_{B \in \partial^+ \cup \partial^-} U_B$, then $i \circ \gamma \cap U = \varnothing$ and $\partial^+ \cup \partial^-/\mathcal{R} \subset \pi(U)$, moreover $\pi^{-1}(\pi(U)) = \pi^{-1}((\partial^+ \cup \partial^-/\mathcal{R}) \cup (U \cap i(M))) = \partial^+ \cup \partial^- \cup (U \cap i(M)) = U$. Thus $\pi(U)$ is a $\mathcal{T}^{\#}/\mathcal{R}$-open neighbourhood of every boundary point, and $i \circ \gamma$ does not enter $\pi(U)$.

(b) If $B \in \partial^+ \cup \partial^- - [P]$ then there is a $\mathscr{T}^{\#}$-open neighbourhood $\tilde{U}_B$ of B and $\exists \delta \in (0, 1)$ such that $\tilde{U}_B \cap i \circ \gamma([\delta, 1)) = \emptyset$. If $U_B := \tilde{U}_B - i \circ \gamma([0, \delta]) - [P]$ and $U := \bigcup_{B \in \partial^+ \cup \partial^- - [P]} U_B$, then $\pi(U)$ is a $\mathscr{T}^{\#}/\mathscr{R}$-open neighbourhood of any $b \in \partial^+ \cup \partial^- / \mathscr{R} - \{\pi(P)\}$ into which $i \circ \gamma$ does not enter. Thus $\pi(P)$ is unique.

Conversely, let $\pi(P)$ be the unique future endpoint of $i \circ \gamma$ in $(M^{\#}/\mathscr{R}, \mathscr{T}^{\#}/\mathscr{R})$. If X is a future endpoint of $i \circ \gamma$ in $(M^{\#}, \mathscr{T}^{\#})$, then $\pi(X)$ is a future endpoint of $i \circ \gamma$ in $(M^{\#}/\mathscr{R}, \mathscr{T}^{\#}/\mathscr{R})$ too. Thus $X \in [P]$. Since $\pi(P)$ is unique, $[P]$ must be closed in $(M^{\#}, \mathscr{T}^{\#})$.

4. The explicit identification and the TIP–TIF gluings

To avoid the difficulties listed in § 1, we would like to have a new explicit identification rule, which is well defined even for strongly causal spacetimes. To ensure that the boundary has the expected and transparent structure, we want to build up $\bar{\mathscr{R}}$ from elementary gluings. Two kinds of elementary identifications are possible: TIP–TIF gluings and TIP–TIP, TIF–TIF gluings. Since the TIP–TIF gluing we will use has been described in the previous paper [7], we only review its basic idea and its most important properties here.

The conformal boundary of non-globally hyperbolic spacetimes (e.g. the Kerr, Reissner–Nordström, anti-de Sitter solutions) have points that are in the chronological future of some points of the spacetime and, at the same time, in the chronological past of others [11, 12, 14, 21]. Within the framework of the causal boundary constructions these boundary points are represented by both a TIP and a TIF. Thus, if we want that the causal boundary of the well known exact solutions are to coincide with their conformal boundary, we have to identify certain TIP with certain TIF.

A TIP P is said to be naked [11] if $P \subset I^-(p)$ for some $p \in M$. If P is a naked TIP, then there is naked TIF F such that, for $\forall q \in F$, $P \subset I^-(q)$ (such a TIF is called a naked counterpart of P) and F is maximal; i.e. F is not a proper subset of any naked counterpart of P [7]. For any $P \in \partial^+$, $\uparrow P = \bigcup_\alpha F_\alpha$, where the F_α are the maximal naked counterparts of P and M is globally hyperbolic iff $\uparrow P = \emptyset$.

The 'ideal points' represented by the terminal irreducible past and future sets are, roughly speaking, the 'top points' and the 'bottom points' of these sets, respectively. Thus it seems reasonable to define the elementary TIP–TIF gluing $\sim$ as follows. For the naked TIP P and the naked TIF F we write $P \sim F$ if they are maximal naked counterparts of each other [7, 10]. The relation $\sim$ is a generalisation of the gluing proposed by Budic and Sachs [5] for causally continuous spacetimes, but $\sim$ does not coincide with that, because $\uparrow P$ is not necessarily a TIF even if the spacetime is causally continuous.

Finally it is worth noting that $\sim$ can be defined for IP and IF too, and there is a deep connection between this redefined $\sim$ and the strong causality condition on M [7].

5. The TIP–TIP, TIF–TIF gluings

As figures 2 and 3 show, there may be different TIP that are not identified by the equivalence relation generated by $\sim$, though they are expected to be identified. This implies that elementary gluings identifying preboundary points of the same kinds should be included as well.

The separation properties of the preboundary points might suggest which preboundary points ought to be identified: the TIP P and P' ($P \neq P'$) are not T_c separated only if $P' \subset P$ and $P' \subset I^-(p)$ for no point p of P, or $P \subset P'$ and $P \subset I^-(p')$ for no point p' of P'. For example, the TIP P and P' in figures 2 and 3 are not T_c separated. The 'ideal points' represented by P' and P are the 'top points' of them. But these 'top points' coincide in figures 2 and 3, and this is just the reason why we want to identify P' and P. Unfortunately, the 'coincidence of the top points of P' and P' is mathematically not well defined. Thus we have to formulate it by means of regular points of P' and P.

For the TIP P' and P we write $P' \approx P$ if, for any $p \in P$ and $q \in P'$, there is a point $r \in P \cap I^+(q)$ such that $I^+(r) \subset I^+(p)$ and $I^-(r) \subset P'$; i.e. $P \cap I^+(q) \cap I^+(p) \cap \bar{P}' \neq \emptyset$ [10]. Trivially, $P' \subset P$ and, due to the strong causality condition, $P' \subset I^-(x)$ for no $x \in P$. Further causal properties of $\approx$ are given by the next statements.

Proposition 5.1. (a) If M is causally continuous then $P' \approx P$ implies $P' = P$.

(b) If $P' \approx P$ and $P' \neq P$, then $P' \neq P$ is not a stable property on P in the sense that, for any generator γ' of P' and any metric $\tilde{g}$ on P being larger than g, $\tilde{P}' := I^-(\gamma'; \tilde{g}) = P$.

Proof. (a) Suppose, on the contrary, that $P' \neq P$. Then $P' \subset P$, $P - \bar{P}' \neq \emptyset$ and let $p \in P - \bar{P}'$. Then for $\forall q \in P'$ there is a point $r \in P \cap I^+(q) \cap \overline{I^+(p)} \cap \bar{P}'$. Thus $I^+(r) \subset I^+(p)$, and, because of the causal continuity, $I^-(p) \subset I^-(r) \subset P'$; i.e. $p \in \bar{P}'$ would follow.

(b) Since $P' \approx P$, there is a point $r \in P \cap I^+(q) \cap \overline{I^+(p)} \cap \bar{P}'$ for $\forall p \in P$, $q \in P'$. If $p \in P - \bar{P}'$ then $r \in \partial P' \cap \partial I^+(p)$ and thus r is a past endpoint of a future endless null geodesic γ lying in $\partial P'$ [13]. Let U be a convex normal neighbourhood of r and let $s \in U \cap (\gamma - \{r\})$. Recall that the metric $\tilde{g}$ on P is larger than the metric g if $g(X, X) \leqslant 0$ implies $\tilde{g}(X, X) < 0$ for any $X \in T_pM$, $p \in P$ [20]. Then $s \in I^+(r; \tilde{g})$, $I^+(r; \tilde{g})$ is open and $s \in \bar{P}' \subset \bar{\tilde{P}}'$; thus $\exists z \in \tilde{P}' \cap I^+(r; \tilde{g})$ implying that $r \in \tilde{P}'$. Since $\tilde{P}'$ is an open neighbourhood of r and $r \in \overline{I^+(p)}$, $\tilde{P}' \cap I^+(p) \neq \emptyset$; i.e. $p \in \tilde{P}'$. Thus $P \subset \tilde{P}'$ and, because of $P' \subset \tilde{P}' \subset P$, $\tilde{P}' = P$.

Thus the relation $\approx$ is trivial for causally continuous spacetimes; and if $\approx$ is not trivial then a special distinguishing instability is present.

Before considering the topological properties of $\approx$, it may be worth noting that $\approx$ can be defined for IP as well and considering how this redefined $\approx$ works on M^-.

Proposition 5.2. If the past-distinguishing condition holds on M, then for the IP P_1, P_2 and any point $z \in M$, $P_1 \approx I^-(z) \approx P_2$ implies $P_1 = I^-(z) = P_2$.

Proof. On the contrary, suppose that $P_1 \approx I^-(z)$ and $P_1 \neq I^-(z)$ for some $z \in M$ and IP P_1. Then $P_1 \subset I^-(z)$ and $z \notin \overline{P_1}$. The manifold topology is regular and thus there are disjoint open neighbourhoods U and V of z and $\overline{P_1}$, respectively. For $\forall p \in I^-(z) \cap U$, $q \in P_1$ there is a point $r \in I^-(z) \cap I^+(q) \cap \overline{I^+(p)} \cap \overline{P_1}$, and a timelike curve λ from r to z. If $x \in \lambda \cap V$, then $x \in I^+(r) \subset I^+(p)$; i.e. there is a timelike curve μ from p to x. Thus μ and the segment of λ between z and x constitute a timelike curve starting at z, leaving U and re-entering U. This, however, contradicts the past-distinguishing condition.

Now suppose that $I^-(z) \approx P_2$ and $I^-(z) \neq P_2$ for some $z \in M$ and IP P_2. If $z \in \overline{I^+(p)}$ for $\forall p \in P_2$, then $z \in \partial P_2$, as otherwise $w \ll w$ would follow for $\forall w \in P_2 \cap I^+(z)$. Let

U_0 be a convex normal neighbourhood of z and γ be a future directed null geodesic from z in ∂P_2. Let $z', z'' \in U_0 \cap (\gamma - \{z\}) \subset \partial P_2$ such that $z' \neq z''$. The past-distinguishing condition holds at z, thus $J^-(z, U) = J^-(z) \cap U$ for some open neighbourhood $U \subset U_0$; moreover, for some $q \in U \cap I^-(z)$, $I^+(q) \cap J^-(z) \cap U = I^+(q, U) \cap J^-(z, U)$ must hold. (If for $\forall q \in U \cap I^-(z)$ there existed a point $x \in I^+(q) \cap J^-(z) \cap U - I^+(q, U) \cap J^-(z, U)$, then there would exist a future directed timelike curve λ starting at q, leaving U and returning to $x \in U$. Since $x \in J^-(z)$, there is a non-spacelike curve μ from x to z. Thus μ and λ would constitute a past directed non-spacelike curve starting at z, leaving U and re-entering U.) This implies that $I^+(q) \cap \overline{I^-(z)} \subset U$, as otherwise the past-distinguishing condition would be violated at z. Because of $I^-(z) \approx P_2$, it follows that, for $\forall p \in P_2$, $\exists r \in P_2 \cap I^+(q) \cap \overline{I^+(p)} \cap \overline{I^-(z)}$. U is contained in a convex normal neighbourhood where $\overline{I^-(z)}$ and $J^-(z)$ coincide, thus $r \in P_2 \cap I^+(q) \cap \overline{I^+(p)} \cap \overline{I^-(z)} \subset P_2 \cap I^+(q) \cap \overline{I^-(z)} \cap U = P_2 \cap I^+(q, U) \cap J^-(z, U)$; i.e. $q \ll r \ll z', z''$. But then $P_2 = I^-(z') = I^-(z'')$ follows, which contradicts the past-distinguishing condition.

Thus $\exists p \in P_2$ for which $z \notin \overline{I^+(p)}$. Then there are disjoint open neighbourhoods U and V of z and $\overline{I^+(p)}$, respectively. For $\forall q \in U \cap I^-(z)$ there is a point $r \in P_2 \cap I^+(q) \cap \overline{I^+(p)} \cap \overline{I^-(z)}$; i.e. there is a timelike curve ν from q to r, and, for $\forall b \in V \cap I^-(r) \cap \nu$, there is a timelike curve μ from b to z. Thus μ and the segment of ν between q and b constitute a past directed timelike curve starting at z, leaving U and re-entering U.

Thus for past-distinguishing spacetimes the redefined $\approx$ is a diagonal relation on the collection $i^-(M)$ of PIP; i.e. $\approx$ does not identify different PIP, or PIP with TIP.

Finally, we have two statements on the separation properties of $\approx$.

Proposition 5.3. If $P_1 \approx P_2$ then P_1 and P_2 are not T_c separated in $\mathcal{T}^{\#}$.

Proof. On the contrary, suppose that there are TIP P_1, P_2 for which $P_1 \approx P_2$ but they are T_c separated. Let γ_1 be a timelike generator of P_1 and let $\{q_m\}$ be a monotonic infinite sequence on γ_1 with no accumulation point in M. Similarly, let γ_2 be a timelike generator of P_2, lying in $P_2 - \overline{P_1}$, and let $\{p_n\}$ be a monotonic infinite sequence on γ_2 without any accumulation point in the manifold topology. Then for $\forall m, n \in N$ $\exists r_{mn} \in P_2 \cap I^+(q_m) \cap \overline{I^+(p_n)} \cap \overline{P_1}$, and let $S := \{r_{mn} | m, n \in N\}$. Of course, $I^-[S] = P_1$ and if $\tilde{S}$ is any subset of S for which $I^-[\tilde{S}] \neq P_1$, then $I^-[\hat{S}] = P_1$ for $\hat{S} := S - \tilde{S}$.

Let U be a $\mathcal{T}^{\#}$-open neighbourhood of P_1. Then, for some point $q' \in P_1$ and IF $F_1, \ldots, F_t$, $P_1 \in V := (I^+(q'))^{\text{int}} \cap F_1^{\text{ext}} \cap \ldots \cap F_t^{\text{ext}} \subset U$. Since $P_1 = I^-[S]$ and $P_1 \in F_1^{\text{ext}} \cap \ldots \cap F_t^{\text{ext}}$, $I^+[S] \not\subset F_i$ must hold for $i = 1, \ldots, t$. Let $\hat{S}_1 := \{s \in S | I^+(s) \not\subset F_1\}$ and $\tilde{S}_1 := S - \hat{S}_1$. Then by definition $I^+[\tilde{S}_1] \subset F_1$, therefore $\tilde{S}_1$ cannot generate P_1, and hence $I^-[\hat{S}_1] = P_1$ and $I^+(s) \not\subset F_1$ for any $s \in \hat{S}_1$. Using a similar argument for $\hat{S}_1$ instead of S, we have a subset $\hat{S}_2$ of S such that $I^-[\hat{S}_2] = P_1$ and $I^+(s) \not\subset F_1 \cup F_2$ for any $s \in \hat{S}_2$. Finally, we have a subset $\hat{S}$ of S such that $I^-[\hat{S}] = P_1$ and $I^+(s) \not\subset F_1 \cup \ldots \cup F_t$ for any $s \in \hat{S}$; i.e. for $\forall s \in \hat{S}$ $\exists w \in I^+(s) - F_1 \cup \ldots \cup F_t$. For this w, $i(w) \in F_1^{\text{ext}} \cap \ldots \cap F_t^{\text{ext}}$.

Let $z \in \gamma_2$. Then $\exists s \in \hat{S}$ for which $I^+(s) \subset I^+(z)$, and for this s there is a point w in $I^+(s) - F_1 \cup \ldots \cup F_t$. Therefore each point of γ_2 is in the chronological past of some point w for that $i(w) \in F_1^{\text{ext}} \cap \ldots \cap F_t^{\text{ext}}$. Thus $i \circ \gamma_2$ cannot leave $F_1^{\text{ext}} \cap \ldots \cap F_t^{\text{ext}}$. Since $q' \in P_1 \subset P_2$, γ_2 enters $I^+(q')$ and $i \circ \gamma_2$ enters $(I^+(q'))^{\text{int}}$; i.e. $i \circ \gamma_2$ enters and remains in $V \subset U$. Therefore P_1 is a $\mathcal{T}^{\#}$ endpoint of γ_2, which contradicts the T_c separation of P_1 and P_2.

Therefore $\approx$ does not identify TIP that are T_c separated in $\mathcal{T}^{\#}$; or, in other words, TIP that are $\approx$-related cannot be T_c separated. We saw in § 3 that the violation of the asymptotic causality condition also creates non-T_c-separated preboundary points. The last statement of the present section tells us that there is no other possibility.

Proposition 5.4. Let the extended strong causality condition hold on $M^{\#}$. If P_0 and P are different TIP such that P_0 is a $\mathcal{T}^{\#}$-future endpoint of some non-spacelike generator γ of P, then $P_0 \approx P$.

Proof. If $V := (I^+(z))^{\text{int}} \cap F_1^{\text{ext}} \cap \ldots \cap F_l^{\text{ext}}$, $z \in P_0$, is a $\mathcal{T}^{\#}$-open neighbourhood of P_0, then, for some $\varepsilon > 0$, $i \circ \gamma([1-\varepsilon, 1)) \subset V$; i.e. $\gamma([1-\varepsilon, 1)) \subset I^+(z) \cap (M - \overline{F_1}) \cap \ldots \cap (M - \overline{F_l})$. This implies that, for $\forall p \in P$, there is a set $S_p \subset M$ such that $I^-[S_p] = P_0$ and $I^+[S_p] \subset I^+(p)$. Let $q \in P_0$ and $S_{pq} := S_p \cap I^+(q)$. Then $P_0 = I^-[S_{pq}]$ and $S_{pq} \subset \overline{P_0} \cap I^+(p) \cap I^+(q)$.

Suppose, on the contrary, that $P_0 \not\approx P$. Then there exist points $p_1 \in P$, $q_1 \in P_0$ for which $P \cap I^+(q_1) \cap I^+(p_1) \cap \overline{P_0} = \varnothing$, implying that $P \cap I^+(q) \cap I^+(p) \cap \overline{P_0} = \varnothing$ for $\forall p \in I^+(p_1) \cap P$, $\forall q \in I^+(q_1) \cap P_0$. Therefore $S_{pq} \cap P = \varnothing$ and, because of $P_0 \subset P$, $S_{pq} \subset \partial P_0 \cap \partial P \cap I^+(q) \cap I^+(p)$. This implies $I^+[S_{pq}] \cap \bar{P} = \varnothing$ and $I^+[S_{pq}] \subset I^+(p)$; i.e. if $x \in I^+[S_{pq}]$ then there is a timelike curve from p to x, which leaves P because $x \in M - \bar{P}$. Therefore the extended strong causality condition is violated at $P_0 \in \partial^+$.

6. The equivalence $\bar{\mathcal{R}}$ and the structure of $(\bar{M}, \bar{\mathcal{T}})$

$\bar{\mathcal{R}}$ is defined as the minimal equivalence relation on $M^{\#}$, being diagonal on $i(M)$, generated by $\sim$ and $\approx$. For all $X \in M^{\#}$ let $(X, X) \in \bar{\mathcal{R}}$, and for $B, B' \in \partial^+ \cup \partial^-$, $B \neq B'$, let $(B, B') \in \bar{\mathcal{R}}$ if for a finite number of preboundary points $B_1, \ldots, B_r$ $B \simeq B_1 \simeq \ldots \simeq B_r \simeq B'$ holds, where $Y \simeq Z$ if $Y \sim Z$, or $Y \approx Z$ or $Z \approx Y$.

$\bar{\mathcal{R}}$ is well defined even for strongly causal spacetimes and, since it is built up from the elementary gluings $\sim$ and $\approx$, the boundary structure is transparent.

In general, the completed spacetime $(\bar{M}, \bar{\mathcal{T}})$ is not a T_1 space, but the inner and boundary points are always T_1 separated [7]. If, in addition, the extended strong causality condition holds on $M^{\#}$, then by the definition of $\bar{\mathcal{R}}$ and by proposition 5.4, $\bar{\mathcal{R}}$ identifies the preboundary points that are not T_c separated in $\mathcal{T}^{\#}$; moreover, because of proposition 3.4(a), inner and boundary points are T_c separated. To get T_c separation for the whole $(\bar{M}, \bar{\mathcal{T}})$, according to proposition 3.4(b), the equivalence classes $[X]$, $X \in M^{\#}$, should be closed in $(M^{\#}, \mathcal{T}^{\#})$. Unfortunately, the $\bar{\mathcal{R}}$-equivalence classes are not necessarily closed in $(M^{\#}, \mathcal{T}^{\#})$. However, in the physically important special case when $\bar{\mathcal{R}}$ is finite the equivalence classes are closed.

The equivalence $\mathcal{R}$ is said to be finite if each set $[X] := \{Z \in M^{\#} | (X, Z) \in \mathcal{R}\}$ has finite elements for $\forall X \in M^{\#}$.

Proposition 6.1. Let the extended strong causality condition hold on $M^{\#}$. If $\bar{\mathcal{R}}$ is finite then it is closed; i.e. $[X]$ is closed in $\mathcal{T}^{\#}$ for any $X \in M^{\#}$.

Proof. Let $X \in i(M)$, i.e. $X = i(p)$ for some $p \in M$. Then, since the inner and preboundary points are always T_1 separated in $\mathcal{T}^{\#}$, $\{i(q)\}$ is closed in $\mathcal{T}^{\#}$ for any $\mathcal{R}$ by proposition 2.2(a) of [7].

Let $X \in \partial^+ \cup \partial^-$. If $q \in M$, then $i(M)$ is a $\mathcal{T}^{\#}$-open neighbourhood of $i(q)$, and $[X] \cap i(M) = \varnothing$. Thus every inner point of $M^{\#}$ has a $\mathcal{T}^{\#}$-open neighbourhood intersecting $[X]$ in an empty set, for any $\mathcal{R}$. Therefore we have to show that each preboundary point not contained in $[X]$ has a $\mathscr{T}^{\#}$-open neighbourhood with an empty intersection with $[X]$.

Let $P_0 \in \partial^+ - [X]$ and

$$\chi_1 := \{P \in [X] \cap \partial^+ | P_0 \subset P\}$$
$$\chi_2 := \{P \in [X] \cap \partial^+ | P_0 \not\subset P\}.$$

Then $\chi_1 \cup \chi_2 = [X] \cap \partial^+$ and $\chi_1 \cap \chi_2 = \emptyset$. Since P_0 is irreducible, $P_0 \not\subset \cup \chi_2$, because otherwise $P_0 = \cup \{P \cap P_0 | P \in \chi_2\}$ would be a proper union of future sets. Therefore $P_0 - \cup \chi_2 \neq \emptyset$ and let $p_0 \in P_0 - \cup \chi_2$. Then $P_0 \in (I^+(p_0))^{\text{int}}$ and $P \cap I^+(p_0) = \emptyset$ for $\forall P \in \chi_2$; i.e. $(I^+(p_0))^{\text{int}} \cap \chi_2 = \emptyset$. Since the extended strong causality condition holds on $M^{\#}$, P_0 and any $P \in \chi_1$ are T_c separated (proposition 5.4). For any $p_0 \in P_0$ and $P \in \chi_1$, $P \in (I^+(p_0))^{\text{int}}$. Therefore, for any $P \in \chi_1$, there must be an IF F such that $P_0 \in F^{\text{ext}}$ and any non-spacelike generator of P leaves F^{ext}. Then, of course, $P \notin F^{\text{ext}}$.

If $\bar{\mathscr{R}}$ is finite, then $[X] \cap \partial^+$, and therefore χ_1 itself contains only finitely many TIP. If $\chi_1 = \{P_1, \dots P_r\}$, then $V := (I^+(p_0))^{\text{int}} \cap F_1^{\text{ext}} \cap \dots \cap F_r^{\text{ext}}$ is a $\mathscr{T}^{\#}$-open neighbourhood of P_0 intersecting $[X] \cap \partial^+$ in an empty set. In a similar way, one can show that each $F_0 \in \partial^- - [X]$ has a $\mathscr{T}^{\#}$-open neighbourhood W such that $W \cap [X] = \emptyset$. Thus $[X]$ is closed in $(M^{\#}, \mathscr{T}^{\#})$.

Thus we have proved the following main statement of the present and the previous papers.

Theorem. If the extended strong causality condition holds on $M^{\#}$ and $\bar{\mathscr{R}}$ is finite, then $(\bar{M}, \bar{\mathscr{T}})$ is a T_c space.

7. Discussion

To get the boundary ∂_c, a topology $\mathscr{T}^{\#}$ and, in the form of an equivalence relation, an identification rule $\bar{\mathscr{R}}$ should be given on $M^{\#}$. They are expected to be defined in terms of causality. To ensure that the structure of M is not to be changed, $\bar{\mathscr{R}}$ should be diagonal on $i(M)$ and $\pi \circ i$ should be an open embedding; and to ensure ∂_c to be boundary in the sense of topology, $\pi \circ i$ should be dense [6]. Beyond these requirements, however, no restriction comes from the basic idea of the construction and therefore one can ask what additional requirements for $\bar{\mathscr{R}}$ and $\mathscr{T}^{\#}$ can be adopted.

The weakest reasonable additional requirement is the claim that the causal boundary of the well known exact solutions are to coincide with their boundary obtained by the conformal technique. If the spacetime can be conformally extended into a larger spacetime, then one could expect the causal boundary to coincide with the boundary (or a part of the boundary) obtained from the extension. This means that there is a spacetime $(\tilde{M}, \tilde{g})$ and a conformal embedding θ of (M, g) into $(\tilde{M}, \tilde{g})$ such that each point b of ∂_c is represented by some point $\tilde{p}$ of $\tilde{M}$ and the neighbourhoods U of b in $(\bar{M}, \bar{\mathscr{T}})$ are expected to have the form $\tilde{U} \cap \overline{\theta(M)}$ for some open neighbourhoods $\tilde{U}$ of $\tilde{p}$ in $\tilde{M}$. (See also [7, 10, 19], but in [7] only a much more restricted form of this requirement was used.) Although this claim seems to be a reasonable requirement, there are two difficulties with this concept. First, if M can be conformally extended into a larger $\tilde{M}$, then, in general, there may be another inequivalent extension into another $\tilde{\tilde{M}}$; but there is no rule to choose from these extensions. Second, since both $\partial^+ \cup \partial^-$ and $\mathscr{T}^{\#}$ are defined by causality, ∂_c may be different from the boundary obtained from the extension even if the extension is unique. For example, Taub's spatially homogenous vacuum solution [12] can be extended into the NUT spaces through the spheres S^3 [14, 22]; its future boundary consists of a single TIP. This deviation of the point-set structure of the boundaries is due to the violation of the past-distinguishing condition at the points of the boundary S^3 in the NUT extension. Unfortunately, similar deviation of the topological structures of the boundaries may occur even if the point-set structure is the expected one. (See, for example, the spacetime shown by figure 1.) Thus for general strongly causal spacetimes neither the topological nor the point-set structure of ∂_c can be expected to coincide with those of the boundary obtained from the extension. It is important to note that these negative results are largely independent of the details of $\bar{\mathscr{R}}$ and $\mathscr{T}^{\#}$; they come from the fact that M is embedded into $(\bar{M}, \bar{\mathscr{R}})$ and M may have certain asymptotic causal pathologies. Thus without a certain asymptotic causality condition one cannot hope for nice separation properties for $(\bar{M}, \bar{\mathscr{T}})$.

The simplest way to obtain a well defined explicit identification rule yielding an acceptable point-set structure for ∂_c, is to build up $\bar{\mathscr{R}}$ from appropriate elementary TIP–TIF and TIP–TIP, TIF–TIF gluings. While the ideas behind these gluings are clear, their mathematical formulation is not so trivial.

Our TIP–TIF gluing, $\sim$, identifies TIP with TIF only on the naked part of $\partial^+ \cup \partial^-$. $\sim$ is a natural generalisation of the explicit identification of Budic and Sachs [5], and it is the extension of the identification of the PIP with the corresponding PIF as well. $\sim$ yields the intuitively expected point-set structure for the spacetimes that are counter-examples to the traditional construction [7-9].

Figure 2 shows a spacetime where the TIP P is 'split up' into different pieces by some cutting. The TIP–TIP, TIF–TIF gluings are intended to identify these different pieces of the TIP and TIF, respectively. Hence one can think that the new gluing $\approx$ is to be related to certain distinguishing properties of M. Our $\approx$ identifies the TIP P and P' in figures 2 and 3, and, in fact, is related to both the distinguishing conditions and the distinguishing instabilities. However, because of the following reasons, $\approx$ does not seem to be as permanent an element of the construction as $\sim$ does. $\sim$ is much more elegant and aesthetic than $\approx$; moreover, figure 4 shows a two-dimensional spacetime, embedded into the two-dimensional Minkowski plane, where the TIP P and P' can be expected to be identified but $\approx$ does not glue them together. Since $\bar{\mathscr{R}}$ does not identify P and P', it does not necessarily yield the point-set structure inherited from the embedding even if both the physical and the embedding spacetimes are causally well behaved. This may imply that $\approx$ should be replaced by new TIP–TIP, TIF–TIF gluings, but the idea we use probably remains the same.

To prove statements for the separation properties of $(\bar{M}, \bar{\mathscr{T}})$, the topology $\mathscr{T}^{\#}$ should be specified. Such is the topology $\mathscr{T}^{\#}_{\mathrm{GKP}}$ [4]. Since $\pi \circ i$ must be an open dense embedding and $\mathscr{T}^{\#}$ must be defined by means of causality, it does not seem to be possible to avoid the occurrence of the neighbourhoods of the preboundary points like that shown by figure 1, which are not 'concentrated on the preboundary points'. Thus it does not seem to be possible to get much better separation properties than that $\mathscr{T}^{\#}_{\mathrm{GKP}}$ yields; and therefore the topology we use is $\mathscr{T}^{\#}_{\mathrm{GKP}}$.

In general, $(\bar{M}, \bar{\mathscr{T}})$ is not necessarily a T_1 space. However, excluding certain asymptotic causal pathologies, one can obtain better separation properties. In the present paper we have concentrated on axiom T_c that ensures unique future and past endpoints for the non-spacelike curves. If the extended strong causality condition holds, then, in terms of $\mathscr{R}$ and $\mathscr{T}^{\#}$, necessary and sufficient conditions can be proved for $(M^{\#}/\mathscr{R}, \mathscr{T}^{\#}/\mathscr{R})$ being a T_c space. We have proved that if the strong causality

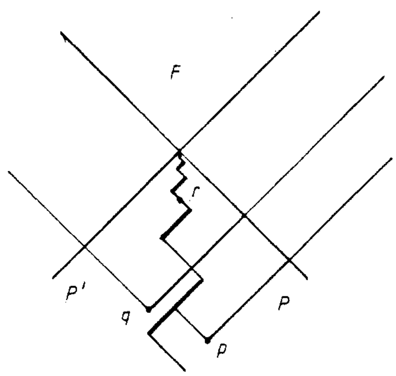

Figure 3. M is the Minkowski plane from which the countable infinity of closed segments and its accumulation point are cut. Although $P \sim F$, P' is not identified with P and F by the equivalence generated by $\sim$. However, $P \approx P'$.

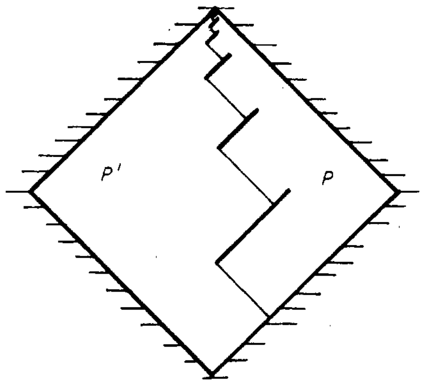

Figure 4. P and P' are not identified by the equivalence $\bar{\mathscr{R}}$ generated by $\sim$ and $\approx$.

condition holds at the preboundary points too, which is an asymptotic causality condition, and the $\bar{\mathscr{R}}$-equivalence classes consist only of finitely many elements, then $(\bar{M}, \bar{\mathscr{T}})$ is a T_c space. The extended strong causality condition is implied by causal continuity, but it does not necessarily imply Carter's second-order strong causality condition. Since $\bar{\mathscr{R}}$ is diagonal on $i(M)$, the finiteness of $\bar{\mathscr{R}}$ means that the causal boundary points of M are not infinitely split up.

For example, the non-spacelike curves of the plane-symmetric static-vacuum solution of Taub [8] have unique future and past endpoints in the completed space, which is not a T_2 space. Finally, it is worth noting that the completed space of the spacetime shown by figure 4, where the point-set structure of the boundary is not the intuitively expected one, is a T_2 space.

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