The Poincaré Structure and the Centre-of-Mass of Asymptotically Flat Spacetimes

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Abstract. The asymptotic symmetries and the conserved quantities of asymptotically flat spacetimes are investigated by extending the canonical analysis of vacuum general relativity of Beig and Ó Murchadha. It is shown that the algebra of asymptotic Killing symmetries, defined with respect to a given foliation of the spacetime, depends on the fall-off rate of the metric. It is only the Lorentz Lie algebra for slow fall-off, but it is the Poincaré algebra for $1/r$ or faster fall-off. The energy-momentum and (relativistic) angular momentum are defined by the value of the Beig–Ó Murchadha Hamiltonian with lapse and shift corresponding to asymptotic Killing vectors. While this energy-momentum and spatial angular momentum reproduce the familiar ADM energy-momentum and Regge–Teitelboim angular momentum, respectively, the centre-of-mass deviates from that of Beig and Ó Murchadha. The new centre-of-mass is conserved, and, together with the spatial angular momentum, form an anti-symmetric Lorentz tensor which transforms just in the correct way under asymptotic Poincaré transformations of the asymptotically Cartesian coordinate system.

1 Introduction

Conserved quantities in various areas of physics play distinguished role, because they reduce the number of equations of motion to solve. In particular, in mechanical systems with only a few degrees of freedom the conserved quantities can (and e.g. in the Kepler problem do) specify the whole dynamics. It is true that in (not completely integrable) field theories they do not, but they can be used to parameterize the solutions of the field equations. In many cases they provide an essential characterization of the states of the physical system. For example, in Newtonian astrophysics the classification of stars is based on their total mass and the total angular momentum with respect to their own centre-of-mass, which classification is essential in the sense that even the qualitative feature of the history of the stars depends critically on the value of these parameters.

Apart from cosmology, both in general relativity and in non-gravitational physics primarily we are interested in localized systems. These systems are

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modeled by appropriately decaying fields near infinity, whenever physical quantities, like total energy-momentum and angular momentum, can be associated with the whole system. However, as is well known, we should make a distinction between null infinity and spatial infinity. If we are interested e.g. in radiative problems then null infinity and physical quantities defined there will have significance. The familiar physical quantities are not conserved in general, rather they change in time characterizing the main aspects of the dynamics, telling us e.g. how much energy is carried away by radiation. (For a recent review see e.g. [1].) On the other hand, if we are interested only in the structure of the theory, e.g. to understand the gauge freedom or the genuine conserved quantities in the theory, then we usually consider decaying at spatial infinity. (For a possible, viable unification of the null and spatial infinities and the connection between these two, see for example [2].) One of the most natural frameworks in which these quantities are introduced is based on the Hamiltonian [3–5]. Several remarkable statements have been proven on their properties [6–8], among which the most important is probably the positive energy theorem [9, 10] and its extensions.

However, the recent investigations of the energy-momentum and (relativistic) angular momentum at the *quasi-local* level raised the question of whether or not these are the "ultimate" expressions that any reasonable quasi-local expression should reproduce at spatial infinity. (For a general discussion of these questions see e.g. [11], and for a recent, potentially promising particular expression for the centre-of-mass see [12, 13].) In fact, a systematic reexamination of these classical results showed that although the energy-momentum and the spatial angular momentum expressions seem to be the "ultimate" ones, the centre-of-mass should probably be completed by an additional (time dependent) term [14].

The main goal of the present contribution is to give a more detailed discussion of those issues of [14] that were not spelled out in detail. In particular, we extend and refine the analysis and the results of Beig and Ó Murchadha [5] on the structure of asymptotically flat spacetimes, and, especially, on the relativistic centre-of-mass. The novelty of the present approach is that we define the total energy-momentum and relativistic angular momentum as the value of the boundary term in the Beig–Ó Murchadha Hamiltonian using the $3+1$ parts of *asymptotic Killing vectors as the lapse and the shift*. This makes it possible to find the correct *explicit* time dependence of the Hamiltonian, yielding the familiar energy-momentum and spatial angular momentum, but the centre-of-mass deviates from the Beig–Ó Murchadha expression by a term which is the linear momentum times the coordinate time. We will see that the angular momentum 4-tensor built from the spatial angular momentum and the corrected centre-of-mass has much better transformation and conservation properties than the previous expressions.

Many questions in connection with the gravitational energy-momentum and (relativistic) angular momentum can be formulated even in connection with the matter fields in Minkowski spacetime too, and it could be interesting and useful to compare the gravitational and the non-gravitational cases. Thus in Sect. 2 we discuss matter fields in Minkowski spacetime, and then, only in Sect. 3, we consider general asymptotically flat spacetimes. That section is devoted to the evolution equations and the boundary conditions. In Sect. 4 we recall the main points of the analysis and results of Beig and Ó Murchadha [5], and we formulate our questions.

The key objects in the present investigations are the asymptotic Killing vectors. These will be introduced and discussed in Sect. 5. In Sect. 6 we return to the discussion of the Beig–Ó Murchadha Hamiltonian, but, instead of the original time independent lapses and shifts, we use the lapse and shift parts of the asymptotic Killing vectors. Finally, in Sect. 7, we define the total energy-momentum and relativistic angular momentum and discuss their transformation and conservation properties. We summarize the main results in Sect. 8.

Although we aimed at giving a logically complete treatise, several important issues, e.g. the discussion of the background (in) dependence of the physical quantities, had to be left out. These can be found in [14]. We consider metrics with faster than $1/r$ fall-off as well. If the conditions of the positive energy theorem are satisfied then these fast fall-off metrics correspond only to flat spacetime configurations. However, in our investigations the 3-space Σ is not assumed to be complete, and its inner boundaries are not assumed to be marginally trapped surfaces. Hence the positive energy theorem does not imply flatness for fast fall-off. Thus it might be worth considering the fast fall-off case as well.

We use the abstract index formalism, and only the underlined and boldface indices take numerical values. The signature of the spacetime metric is -2, and the Riemann and Ricci tensors and the curvature scalar e.g. of the spacetime covariant derivative ∇_e will be defined by $-{}^4R^a{}_{bcd}X^bY^cZ^d := \nabla_Y(\nabla_Z X^a) - \nabla_Z(\nabla_Y X^a) - \nabla_{[Y,Z]}X^a$, ${}^4R_{ab} := {}^4R^c{}_{acb}$ and ${}^4R := {}^4R_{ab}g^{ab}$, respectively. Thus Einstein's equations take the form ${}^4G_{ab} := {}^4R_{ab} - \frac{1}{2}{}^4Rg_{ab} = -\Lambda g_{ab} - \kappa T_{ab}$, and we use the units in which $c = 1$.

2 Symmetries and Conserved Quantities in Minkowski Spacetime

2.1 The Killing Fields of the Minkowski Spacetime

It is well known that the Killing vectors of the Minkowski spacetime form a ten dimensional Lie algebra $\mathcal{K}$, which contains a four dimensional commutative ideal $\mathcal{T}$, and the quotient $\mathcal{K}/\mathcal{T}$ is isomorphic to $so(1,3)$. The elements of $\mathcal{T}$ are the *constant* vector fields, called the translations, which inherit a natural Lorentzian metric from g_{ab}. If a point o of the Minkowski spacetime is fixed, then the quotient $\mathcal{K}/\mathcal{T}$ can be identified as the Lie algebra of those

Killing fields that are vanishing at o: They are the rotation-boost Killing vectors. Thus while the ideal of the constant vector fields is canonically determined by the geometric structure of the spacetime, the quotient $\mathcal{K}/\mathcal{T}$ can be realized by Killing fields only if the 'origin' o has been specified.

If an orthonormal basis $\{E^a_{\underline{a}}\}$, $\underline{a} = 0, \dots, 3$, of *constant* vector fields and the "origin" o have been chosen, then the familiar Cartesian coordinate system $\{x^{\underline{a}}\}$ is fixed by $E^a_{\underline{a}} = (\partial/\partial x^{\underline{a}})^a$ and $x^{\underline{a}}(o) = 0$. (Underlined Roman indices from the beginning of the alphabet are concrete *spacetime name* indices.) Thus this is not only a coordinate system in the sense of differential topology, but it has a metrical content as well. Obviously, if we change the vector basis by a Lorentz transformation, $E^a_{\underline{a}} \mapsto E^a_{\underline{b}} \Lambda^{\underline{b}}{}_{\underline{a}}$, and the origin o is shifted to a new point, then the Cartesian coordinates change according to the Poincaré transformation: $x^{\underline{a}} \mapsto x^{\underline{b}} \Lambda_{\underline{b}}{}^{\underline{a}} + C^{\underline{a}}$, where $\Lambda^{\underline{c}}{}_{\underline{b}} \Lambda_{\underline{c}}{}^{\underline{a}} = \delta^{\underline{a}}_{\underline{b}}$, and $C^{\underline{a}} \in \mathbb{R}^4$ characterizes the shift of the origin.

If the basis vector $E^a_{\mathbf{0}}$ is future pointing and timelike, then we usually write the Cartesian coordinates as $x^{\underline{a}} = (t, x^{\mathbf{i}})$, $\mathbf{i} = 1, 2, 3$. Thus the boldface Roman indices from the middle of the alphabet are concrete *spatial name* indices. In a fixed Cartesian coordinate system the general form of a Killing 1-form, given both in its covariant and its $3+1$ forms, is

$$\begin{aligned} K_a &= T_{\underline{a}} \nabla_a x^{\underline{a}} + M_{\underline{a}\,\underline{b}} \left(x^{\underline{a}} \nabla_a x^{\underline{b}} - x^{\underline{b}} \nabla_a x^{\underline{a}} \right) \\ &= \left(2x^{\mathbf{k}} M_{\mathbf{ki}} + T_{\mathbf{i}} - 2t M_{\mathbf{i0}} \right) \nabla_a x^{\mathbf{i}} + \left(2x^{\mathbf{k}} M_{\mathbf{k0}} + T_{\mathbf{0}} \right) \nabla_a t. \end{aligned} \tag{1}$$

This is a linear combination of the independent translation and rotation-boost Killing 1-forms, $K^{\underline{a}}_a := \nabla_a x^{\underline{a}}$ and $K^{\underline{a}\,\underline{b}}_a := x^{\underline{a}} \nabla_a x^{\underline{b}} - x^{\underline{b}} \nabla_a x^{\underline{a}}$, respectively, by constant coefficients $T_{\underline{a}}$ and $M_{\underline{a}\,\underline{b}} = -M_{\underline{b}\,\underline{a}}$. $T_{\mathbf{0}}$, $T_{\mathbf{i}}$, $M_{\mathbf{ij}}$ and $M_{\mathbf{i0}}$ are the components of the time and space translations, and the rotation and boost parts of K_a, respectively, in the coordinates $\{x^{\underline{a}}\}$. Note that the spatial components (in the $3+1$ form) of the boost Killing 1-forms depend linearly not only on the spatial coordinates, but on the Cartesian time coordinate as well.

2.2 Quasi-Local Energy-Momentum and Angular Momentum

Let Σ be any smooth, compact, spacelike hypersurface with smooth boundary $\mathcal{S} := \partial\Sigma$. If t^a is its future directed unit timelike normal, $\mathrm{d}\Sigma$ is the induced volume element on Σ and T^{ab} is the energy-momentum tensor of the matter fields, then we can form the flux integrals

$$\mathtt{Q}^m_\Sigma[K^a] := \int_\Sigma K_a T^{ab} t_b \mathrm{d}\Sigma \tag{2}$$

for any vector field K^a. If, however, K^a is a Killing vector, then $\mathtt{Q}^m_\Sigma[K^a]$ is conserved in the sense that if Σ' is another compact spacelike hypersurface with the same boundary $\mathcal{S}$, then the flux integrals defined on Σ and Σ'

coincide. In particular, if $D(\Sigma)$ is the domain of dependence of Σ and ξ^a is a "general time axis" compatible with a foliation Σ_t of $D(\Sigma)$ (in the sense that the Lie dragging of one leaf of the foliation along the integral curves of ξ^a with a given parameter value is another leaf), then the Lie derivative of $\mathtt{Q}^m_{\Sigma_t}[K^a]$ along ξ^a is vanishing provided that K^a is a Killing field. Therefore, for Killing vectors K^a the flux integral (2) is in fact associated with the closed spacelike 2-surface $\mathcal{S}$: $\mathtt{Q}^m_{\Sigma}[K^a] = \mathtt{Q}^m_{\mathcal{S}}[K^a]$. Note that the lapse function N of the foliation Σ_t is vanishing on $\mathcal{S}$, and the shift vector N^a is tangent to $\mathcal{S}$ on $\mathcal{S}$. The "general time axis" ξ^a need not be timelike or related to the symmetry generators K^a in any way.

Since $\mathtt{Q}^m_{\mathcal{S}}[K^a]$ is linear in K_a, by (1) in a fixed Cartesian coordinate system it has the structure $\mathtt{Q}^m_{\mathcal{S}}[K^a] = T_{\underline{a}}\,\mathtt{P}^{\underline{a}} + M_{\underline{a}\,\underline{b}}\,\mathtt{J}^{\underline{a}\,\underline{b}}$. The coefficients of the parameters $T_{\underline{a}}$ and $M_{\underline{a}\,\underline{b}}$ define the quasi-local energy-momentum and (relativistic) angular momentum of the matter fields, respectively, associated with the closed spacelike 2-surface $\mathcal{S}$. If $\mu := T^{ab}t_a t_b$ and $j^a := P^a_b T^{bc} t_c$ are the energy-density and the momentum density of the matter fields seen by the observer t^a, where $P^a_b := \delta^a_b - t^a t_b$ is the orthogonal projection to Σ, then these quasi-local quantities can be given explicitly in terms of the independent translation and rotation-boost Killing vectors as

$$\mathtt{P}^{\underline{a}} = \int_\Sigma K^{\underline{a}}_a \,(\mu t^a + j^a)\,\mathrm{d}\Sigma, \qquad \mathtt{J}^{\underline{a}\,\underline{b}} = \int_\Sigma K^{\underline{a}\,\underline{b}}_a \,(\mu t^a + j^a)\,\mathrm{d}\Sigma\,. \tag{3}$$

(For a more detailed discussion of these concepts see e.g. [11].) These integrals depend on the choice for the Cartesian coordinate system, but it is easy to see that under the Poincaré transformation $x^{\underline{a}} \mapsto x^{\underline{b}}\,\Lambda_{\underline{b}}{}^{\underline{a}} + C^{\underline{a}}$ of the coordinates $\mathtt{P}^{\underline{a}}$ and $\mathtt{J}^{\underline{a}\,\underline{b}}$ transform just in the expected correct way: $\mathtt{P}^{\underline{a}} \mapsto \mathtt{P}^{\underline{b}}\,\Lambda_{\underline{b}}{}^{\underline{a}}$ and $\mathtt{J}^{\underline{a}\,\underline{b}} \mapsto \mathtt{J}^{\underline{c}\,\underline{d}}\,\Lambda_{\underline{c}}{}^{\underline{a}}\,\Lambda_{\underline{d}}{}^{\underline{b}} + \mathtt{P}^{\underline{c}}\,(C^{\underline{a}}\,\Lambda_{\underline{c}}{}^{\underline{b}} - C^{\underline{b}}\,\Lambda_{\underline{c}}{}^{\underline{a}})$. Note that, as a consequence of the special linear time dependence of the boost Killing fields in (1), the centre-of-mass part $\mathtt{J}^{\mathbf{i}0}$ of the angular momentum also depends on the Cartesian time coordinate. Without this time dependence it would not be conserved and would not have the correct transformation properties.

2.3 Total Energy-Momentum and Angular Momentum

The flux integral (2) can be defined even if Σ is not compact, e.g. if it extends to spatial infinity of the Minkowski spacetime, provided the integral exists. To ensure the finiteness of this integral, i.e. to have finite total energy-momentum and (relativistic) angular momentum given by (3), certain boundary conditions must be imposed on the energy-density μ and momentum density j^a on Σ. Such a boundary condition e.g. on a $t=$ const hyperplane in the Cartesian coordinates $x^{\underline{a}} = (t, x^{\mathbf{i}})$ might be the *fall-off conditions*

$$\mu = \frac{1}{r^4}\mu^{(4)}\Big(t, \frac{x^{\mathbf{k}}}{r}\Big) + o\big(r^{-4}\big)\,, \tag{4}$$

$$j^{\mathbf{i}} = \frac{1}{r^4} j^{\mathbf{i}(4)}\Big(t, \frac{x^{\mathbf{k}}}{r}\Big) + o\big(r^{-4}\big)\,, \tag{5}$$

for some functions $\mu^{(4)}$ and $j^{\mathbf{i}(4)}$, where $r^2 := \delta_{\mathbf{ij}} x^{\mathbf{i}} x^{\mathbf{j}}$, the square of the radial distance in the hyperplane Σ, and $o(r^{-k})$ denotes a function $f(r)$ for which $\lim_{r\to\infty}(r^k f(r)) = 0$. $o(r^{-0})$ denotes logarithmic fall-off and $o(r^{+0})$ logarithmic divergence. We will use $O(r^{-k})$ to denote a function $f(r)$ for which the limit $\lim_{r\to\infty}(r^k f(r))$ exists. These fall-off conditions ensure the finiteness of the total energy-momentum, but the angular momentum is still diverging logarithmically. Thus to have finite total angular momentum as well, stronger or additional conditions must be imposed. One apparently natural condition could be to require slightly faster than $1/r^4$ fall-off in (4) and (5). Since, however, the typical fall-off rate of the energy and momentum densities of the electromagnetic field is $1/r^4$, by a faster fall-off condition we would exclude the electromagnetic field from our investigations. Thus we retain the $1/r^4$ fall-off, and seek for additional conditions.

Evaluating the total angular momentum expression with the energy and momentum densities satisfying (4) and (5), one arrives at the additional necessary and sufficient conditions

$$\oint_{\mathcal{S}} v^{[\mathbf{i}} j^{\mathbf{j}]} \, \mathrm{d}\mathcal{S}_1 = o\big(r^{-4}\big) \, , \tag{6}$$

$$\oint_{\mathcal{S}} v^{\mathbf{i}} \mu \, \mathrm{d}\mathcal{S}_1 = o\big(r^{-4}\big) \, . \tag{7}$$

Here v^a is the outward directed unit normal to the large sphere $\mathcal{S}$ of radius r in the hyperplane Σ, and $\mathrm{d}\mathcal{S}_1$ is the area element on the *unit* sphere. However, the *global integral conditions* (6)–(7) are only implicit restrictions on the asymptotic behaviour of μ and j^a, and hence it is difficult to use them in practice. If we are not interested in the exact boundary conditions, as in the present discussion, then we prefer to have only an *explicitly given* sufficient condition. Such a sufficient condition might be the global *parity condition*: The leading terms in (4) and (5) are required to be even parity functions of their second argument: $\mu^{(4)}(t, \frac{x^{\mathbf{k}}}{r}) = \mu^{(4)}(t, -\frac{x^{\mathbf{k}}}{r})$ and $j^{\mathbf{i}(4)}(t, \frac{x^{\mathbf{k}}}{r}) = j^{\mathbf{i}(4)}(t, -\frac{x^{\mathbf{k}}}{r})$. Then the fall-off and parity conditions together ensure the finiteness of the total energy-momentum and (relativistic) angular momentum of the matter fields.

It is easy to check that if the fall-off and parity conditions above are imposed not only on a single spacelike hyperplane but on boosted hyperplanes as well, then the spatial stress part of the energy-momentum tensor, $\sigma^{ab} := P^a_c P^b_d T^{cd}$, must also have the asymptotic structure

$$\sigma^{\mathbf{ij}} = \frac{1}{r^4} \sigma^{\mathbf{ij}(4)}\Big(t, \frac{x^{\mathbf{k}}}{r}\Big) + o\big(r^{-4}\big) \, , \tag{8}$$

and the leading term $\sigma^{\mathbf{ij}(4)}(t, \frac{x^{\mathbf{k}}}{r})$ must be an even parity function of $\frac{x^{\mathbf{k}}}{r}$.

2.4 Asymptotically Cartesian Coordinate Systems

By the results of the previous two subsections the Cartesian coordinates appear to play a fundamental role in the definition and the study of the properties of the conserved quantities in Minkowski spacetime. But as we saw in Subsect. 2.1, the Cartesian coordinates have metrical content, because, by their very definition, they are adapted to exact geometric symmetries of the spacetime. However, primarily we are interested in general, non-flat asymptotically flat spacetimes, where we do not have any exact geometric symmetry. Thus the question arises naturally whether or not there is some natural generalization of the familiar Cartesian coordinates, at least asymptotically, even in a general asymptotically flat spacetime, which could play an analogous role in constructing the conserved quantities.

Such an asymptotically Cartesian coordinate system $(\tau, \eta^{\mathbf{i}})$ may be based on a foliation Σ_τ of the asymptotically flat spacetime, which foliation can be characterized on a typical leaf Σ of the foliation by the lapse function N. Furthermore, we need to have a shift vector N^a as well, which tells us how the spatial Cartesian coordinates $\eta^{\mathbf{i}}$, introduced on one leaf of the foliation, is extended to the neighbouring leaves. Thus we would like to find a criterion, formulated in terms of the lapse and the shift, when to consider the corresponding coordinate system $(\tau, \eta^{\mathbf{i}})$ to be asymptotically Cartesian.

In Minkowski spacetime the lapse of the Cartesian coordinate system is the constant function with value 1, and the shift is identically vanishing. Therefore, it seems natural to consider the coordinate system $(\tau, \eta^{\mathbf{i}})$ to be asymptotically Cartesian only if $N \to 1$ and $N^a \to 0$ at infinity uniformly, independently of the direction in which the limit is taken on Σ. This naive criterion can also be supported by a formal analysis of the coordinate systems in the conformally compactified Minkowski spacetime near the spatial infinity i^0 [14]: There exists a flat metric ${}_0q_{ab}$ on Σ such that $q_{\mathbf{ij}} - {}_0q_{\mathbf{ij}}$ and $\chi_{\mathbf{ij}}$, the components of the difference of the induced and the flat metrics and of the extrinsic curvature in the ${}_0q_{ab}$–Cartesian coordinates $\eta^{\mathbf{i}}$, respectively, tend to zero as $R^2 := \delta_{\mathbf{ij}}\eta^{\mathbf{i}}\eta^{\mathbf{j}}$ tends to infinity, and moreover $N(\tau, \eta^{\mathbf{k}}) = 1 + O(R^{-1})$ and $N^{\mathbf{i}}(\tau, \eta^{\mathbf{k}}) = O(R^{-1})$. In fact, an asymptotically vanishing N would correspond to a foliation in which the time separation of the different leaves tends to zero, while an asymptotically diverging N would correspond to one in which this time separation is diverging. Thus, in particular, in Minkowski spacetime the coordinate transformation connecting the Cartesian coordinate system to a system $(\tau, \eta^{\mathbf{i}})$ based on an asymptotically vanishing lapse is getting to be singular, i.e. $(\tau, \eta^{\mathbf{i}})$ is "collapsing" asymptotically.

If $(\tau, \eta^{\mathbf{i}})$ is an asymptotically Cartesian coordinate system in Minkowski spacetime based on a smooth spacelike Cauchy surface Σ extending to the spatial infinity, then the Killing field (1) takes the form

$$K_e = M_{\mathbf{ij}}\big(\eta^{\mathbf{i}} D_e \eta^{\mathbf{j}} - \eta^{\mathbf{j}} D_e \eta^{\mathbf{i}}\big) + 2M_{\mathbf{i0}}\big(\eta^{\mathbf{i}}\tau_e - \tau D_e \eta^{\mathbf{i}}\big) + s_{\mathbf{i}} D_e \eta^{\mathbf{i}} + s\tau_e \,. \tag{9}$$

Here D_e is the intrinsic derivative operator, τ_e the future pointing unit timelike normal to Σ and $s(\tau, \eta^{\mathbf{k}}) = s^{(0)}(\tau, \frac{\eta^{\mathbf{k}}}{R}) + O(R^{-1})$, $s_{\mathbf{i}}(\tau, \eta^{\mathbf{k}}) = s^{(0)}_{\mathbf{i}}(\tau, \frac{\eta^{\mathbf{k}}}{R}) + O(R^{-1})$. Thus s and $s_e := s_{\mathbf{i}} D_e \eta^{\mathbf{i}}$, which would be the time and space translation parts of K_e in (1), respectively, depend on τ, $\eta^{\mathbf{i}}$ and $1/R$. Therefore, they are analogous to the supertranslations of the cuts of future null infinity, and the proper translations correspond only to special supertranslations. We will see in Subsects. 3.2 and 4.1 that these are precisely the $\eta^{\mathbf{i}}$-independent supertranslations, while those that are odd parity functions of $\frac{\eta^{\mathbf{k}}}{R}$ are the proper supertranslations and have only gauge content.

2.5 Conservation Properties

We saw in Subsect. 2.2 that the quasi-local energy-momentum and angular momentum are conserved with respect to a time evolution characterized by a vector field ξ^a if the evolution preserves $D(\Sigma)$, i.e. the lapse part of ξ^a is vanishing on $\mathcal{S}$ and the shift part is tangent to $\mathcal{S}$ on $\mathcal{S}$. In the present subsection we formulate the analogous question for the total energy-momentum and angular momentum.

Thus let Σ_τ be a foliation of the Minkowski spacetime by smooth Cauchy surfaces, let t^a be its future pointing unit timelike normal and N the lapse of the foliation. Let N^a be the shift vector and define the "general time axis" $\xi^a := N t^a + N^a$. Then we can take the integrals (3) defining the total energy-momentum and angular momentum on the leaves Σ_τ and calculate their Lie derivative along ξ^a. Our question is what asymptotic conditions should the lapse and the shift satisfy such that these Lie derivatives be vanishing. However, this analysis consists of two things. The first is that even though the integral (3) on a specific hypersurface is finite, it is not necessarily finite on the hypersurfaces obtained by "time evolution" along ξ^a; i.e. *we should ensure that the boundary conditions ensuring the finiteness of (3) be preserved.* The second is to ensure that these finite integrals be the same.

Nevertheless, this analysis can be, and in the next section will be, carried out even in general asymptotically flat spacetimes with vector fields K^a having the asymptotic structure more general than (9). We will see that the total energy-momentum and (relativistic) angular momentum are conserved even if N and N^a are linearly diverging (see (21)–(22)).

3 Asymptotically Flat Spacetimes

3.1 The Boundary Conditions

The definition of the asymptotic flatness of a spacetime that we adopt in the present paper is probably the oldest one. We say that a spacetime is asymptotically flat at spatial infinity if it contains an asymptotically flat spacelike hypersurface Σ. Thus we should define the asymptotic flatness of such a hypersurface. We say that the spacelike hypersurface Σ is (k,l)-asymptotically flat, if (1) there is a (negative definite) background metric ${}_0q_{ab}$ on Σ, which is flat outside a large compact subset $K \subset \Sigma$ such that $\Sigma - K$ is diffeomorphic to $\mathbb{R}^3$ minus a solid ball; (2) for some positive k and l the components $q_{\mathbf{ij}}$ and $\chi_{\mathbf{ij}}$ of the physical induced metric and of the extrinsic curvature, respectively, in the ${}_0q_{ab}$-Cartesian coordinate system on $\Sigma - K$ satisfy the fall-off conditions

$$q_{\mathbf{ij}} - {}_0q_{\mathbf{ij}} = \frac{1}{r^k} q^{(k)}_{\mathbf{ij}} + o\bigl(r^{-k}\bigr), \tag{10}$$

$$\chi_{\mathbf{ij}} = \frac{1}{r^l} \chi^{(l)}_{\mathbf{ij}} + o\bigl(r^{-l}\bigr) \ ; \tag{11}$$

and, (3) the leading terms $q^{(k)}_{\mathbf{ij}}$ and $\chi^{(l)}_{\mathbf{ij}}$ are even and odd parity functions of $\frac{x^{\mathbf{k}}}{r}$, respectively. Here r is the radial coordinate defined by $r^2 := \delta_{\mathbf{ij}} x^{\mathbf{i}} x^{\mathbf{j}}$. In general these conditions do not imply that every component e.g. of the derivative ${}_0D_c q_{ab}$ tends to zero as $1/r^{k+1}$, where ${}_0D_c$ is the derivative operator determined by the background metric, which would be a useful property in practice. Similarly, still not every component of ${}_0D_c\chi_{ab}$ tends to zero as $1/r^{l+1}$. If, however, we assume that the "rests" $m_{ab} := q_{ab} - {}_0q_{ab} - r^{-k} q^{(k)}_{ab}$ and $k_{ab} := \chi_{ab} - r^{-l}\chi^{(l)}_{ab}$ also satisfy

$$ {}_0D_c m_{ab} = o\bigl(r^{-k-1}\bigr),\ \ {}_0D_{d\,0}D_c m_{ab} = o\bigl(r^{-k-2}\bigr),\ \ \dots \tag{12}$$

$$ {}_0D_c \chi_{ab} = o\bigl(r^{-l-1}\bigr),\ \ {}_0D_{d\,0}D_c \chi_{ab} = o\bigl(r^{-l-2}\bigr),\ \ \dots \tag{13}$$

then, together with (10) and (11), these imply ${}_0D_{e_1} \dots {}_0D_{e_s} q_{ab} = O(r^{-k-s})$ and ${}_0D_{e_1} \dots {}_0D_{e_s} \chi_{ab} = O(r^{-l-s})$ for any $s = 1, 2, \dots$, and the parity of these derivatives is $(-)^s$ and $(-)^{s+1}$, respectively. The properties $m_{ab} = o(r^{-k})$, ${}_0D_e m_{ab} = o(r^{-k-1})$, … of m_{ab} will be denoted by $m_{ab} = o^\infty(r^{-k})$. Although it would be enough to require ${}_0D_{e_1} \dots {}_0D_{e_s} m_{ab} = o(r^{-k-s})$ only for some finite s depending on the order of the derivatives that appears in the actual calculations, for the sake of simplicity we assume that $m_{ab} = o^\infty(r^{-k})$. Similarly, we require that $k_{ab} = o^\infty(r^{-l})$.

We assume that the matter fields satisfy boundary conditions that yield energy density μ, momentum density j^a and spatial stress σ^{ab} satisfying the fall-off and parity conditions that we discussed in Subsect. 2.3, defined with respect to the ${}_0q_{ab}$-Cartesian coordinate system. Furthermore, again by technical reasons, we assume that the "rests" appearing in (4), (5) and (8) are also $o^\infty(r^{-4})$. Then we can form the integral

$$\mathtt{Q}^m\bigl[M, M^a\bigr] := \int_\Sigma \bigl(M t_a + M_a\bigr) T^{ab} t_b \mathrm{d}\Sigma \ , \tag{14}$$

and, as a consequence of the boundary conditions for μ and j^a, this integral exists if the asymptotic form of M and M^a is given by

$$M\big(t,x^{\mathbf{k}}\big)=r^A M^{(A)}\Big(t,\frac{x^{\mathbf{k}}}{r}\Big)+o^\infty\big(r^A\big), \tag{15}$$

$$M_{\mathbf{i}}\big(t,x^{\mathbf{k}}\big)=r^B M_{\mathbf{i}}^{(B)}\Big(t,\frac{x^{\mathbf{k}}}{r}\Big)+o^\infty\big(r^B\big)\ , \tag{16}$$

where $A,B\leq 1$ and if the equality holds in these inequalities then $M(t,\frac{x^{\mathbf{k}}}{r})$ and $M_{\mathbf{i}}(t,\frac{x^{\mathbf{k}}}{r})$, respectively, must be odd parity functions of $\frac{x^{\mathbf{k}}}{r}$. Note that (9), and hence (1) also, are special cases of (15)–(16). In the next subsection we discuss the time dependence of $\mathtt{Q}^m[M,M^a]$.

3.2 The Evolution Equations

Let the spacetime be foliated by smooth spacelike Cauchy hypersurfaces Σ_t, and let a "general time axis" $\xi^a=Nt^a+N^a$ be also given. Then the $3+1$ form of the equation $T^{ab}{}_{;b}=0$ is well known to be

$$\dot\mu=N\Big(-D_aj^a+\sigma^{ab}\chi_{ab}-\frac{2}{N}j^aD_aN-\mu\chi\Big)+\text{Ł}_{\mathbf{N}}\mu, \tag{17}$$

$$\dot j^b=N\Big(-D_a\sigma^{ab}-\frac{1}{N}\sigma^{ba}D_aN+\mu\frac{1}{N}D^bN-2j_a\chi^{ba}-\chi j^b\Big)+\text{Ł}_{\mathbf{N}}j^b\ , \tag{18}$$

where the dot denotes the projection to the leaves of the foliation of the Lie derivative along ξ^a. They describe the evolution of the energy density and the momentum density of the matter fields along the integral curves of ξ^a. Similarly, the evolution equations for the geometry are

$$\dot q_{ab}=2N\chi_{ab}+\text{Ł}_{\mathbf{N}}q_{ab}, \tag{19}$$

$$\begin{aligned}\dot\chi_{ab}&=N\Big(-R_{ab}+2\chi_{ac}\chi^c{}_b-\chi\chi_{ab}\Big)+\text{Ł}_{\mathbf{N}}\chi_{ab}-D_aD_bN\\&\quad+\Lambda Nq_{ab}+\kappa N\Big(-\sigma_{ab}+\frac{1}{2}\sigma^e{}_eq_{ab}+\frac{1}{2}\mu q_{ab}\Big)\ .\end{aligned} \tag{20}$$

The first is a simple consequence of the definitions, but the second is the space-space projection of the Einstein equations.

Next suppose that the spacetime is asymptotically flat (whenever the cosmological constant Λ is zero), and characterize the foliation and the general time axis on a typical Cauchy surface Σ by the lapse N and the shift N^a. In the previous subsection we defined the asymptotic flatness of the spacetime by the existence of an appropriately defined asymptotically flat spacelike hypersurface. However, the existence of such a single hypersurface does not imply that the evolution of such a hypersurface will be asymptotically flat, i.e. the boundary conditions are not necessarily preserved by the dynamical equations. Thus our question is what conditions should we impose on the lapse and the shift such that the evolution equations (17)–(20) preserve the fall-off and parity conditions, both for the matter fields and the geometry.

Assuming that the lapse and the shift have the a priori asymptotic form $N(t,x^{\mathbf{k}})=r^CN^{(C)}(t,\frac{x^{\mathbf{k}}}{r})+o^\infty(r^C)$ and $N_{\mathbf{i}}(t,x^{\mathbf{k}})=r^DN_{\mathbf{i}}^{(D)}(t,\frac{x^{\mathbf{k}}}{r})+o^\infty(r^D)$

for some C and D, we can evaluate the right hand side of the evolution equations. If we require that the leading orders and parities on both sides coincide, we obtain two results. The first is a link between the fall-off rates for the metric and the extrinsic curvature: In the generic case $l = k + 1$. (For the exceptional cases see [14].) The other is the detailed asymptotic structure of the lapse and the shift, given by

$$N\big(t, x^{\mathbf{k}}\big) = 2x^{\mathbf{k}}\beta_{\mathbf{k}}\big(t\big) + \tau\big(t\big) + r^E\nu^{(E)}\Big(t, \frac{x^{\mathbf{k}}}{r}\Big) + o^\infty\big(r^E\big), \tag{21}$$

$$N_{\mathbf{i}}\big(t, x^{\mathbf{k}}\big) = 2x^{\mathbf{k}}\rho_{\mathbf{ki}}\big(t\big) + \tau_{\mathbf{i}}\big(t\big) + r^F\nu_{\mathbf{i}}^{(F)}\Big(t, \frac{x^{\mathbf{k}}}{r}\Big) + o^\infty\big(r^F\big)\ . \tag{22}$$

Here the coefficients $\beta_{\mathbf{k}}(t)$, $\tau(t)$, $\rho_{\mathbf{ki}}(t)$ and $\tau_{\mathbf{i}}(t)$ are arbitrary functions of t, the powers E and F are bounded from above by the fall-off rate of the metric: $E, F \leq (1 - k)$, and if the equality holds in these inequalities then the functions $\nu^{(E)}(t, \frac{x^{\mathbf{k}}}{r})$ and $\nu_{\mathbf{i}}^{(F)}(t, \frac{x^{\mathbf{k}}}{r})$ are odd parity functions of their second argument, respectively.

Since the structure of $N_{\mathbf{i}}$ is similar to that of N, it is enough to discuss only e.g. (21). By $k > 0$ the leading term in (21) is the first, but to decide whether the next order is the second or the third, we should consider the disjoint cases $k > 1$, $k < 1$ and $k = 1$. If $k > 1$, which corresponds to a fast fall-off metric, then the third term tends to zero at infinity as r^E, where E is *negative*, whenever the next order term is the second. If $k < 1$, which corresponds to a slow fall-off, E may be positive, and if E is actually positive, then the third term is diverging. In this case there is no reason to keep the second term, because that cannot be isolated in the presence of the uncontrollable diverging term. If $k = 1$, then E may be zero, and if it is actually zero, then both the second and the third terms are asymptotically of the same order. However, in spite of the fact that the third term is uncontrollable and of the same order asymptotically as the second, we can make a natural distinction between these: The second, being independent of the spatial coordinates, is an *even* parity, while the third is an *odd* parity function of $\frac{x^{\mathbf{k}}}{r}$. Thus the structure of N and N^a resembles the structure of the timelike and spacelike projection of the Killing fields of the Minkowski spacetime given by (1) or rather (9). In particular, $\beta_{\mathbf{k}}(t)$, $\rho_{\mathbf{ki}}(t)$, $\tau(t)$ and $\tau_{\mathbf{i}}(t)$ are analogous to the boost, rotation, time translation and spatial translation generators, and the terms $r^E\nu^{(E)}$ and $r^F\nu_{\mathbf{i}}^{(F)}$ are similar to the proper temporal and spatial supertranslations of (9). However, while the components of the Killing fields have a special time dependence, the parameters $\beta_{\mathbf{k}}(t)$, $\rho_{\mathbf{ki}}(t)$, $\tau(t)$ and $\tau_{\mathbf{i}}(t)$ may have arbitrary time dependence.

Defining the integral $\mathtt{Q}^m[M, M^a]$, given by (14), on each of the leaves Σ_t of the foliation, one can compute its time derivative. It is

$$\dot{\mathtt{Q}}^m\left[M, M^a\right] = \int_{\Sigma_t} \Big(\mu\big(\dot{M} + M^a D_a N - N^a D_a M\big) \\ + j_a\big(\dot{M}^a + N D^a M - M D^a N - [N, M]^a\big) \\ + \sigma_{ab} N\big(M\chi^{ab} + D^{(a}M^{b)}\big) \\ + D_a\Big(\big(\mu M + j_b M^b\big)N^a - \big(j^a M + \sigma^{ab} M_b\big)N\big)\Big)\mathrm{d}\Sigma_t \; . \quad (23)$$

Taking into account the boundary conditions and substituting the asymptotic form (21)–(22) here we find that $\dot{\mathtt{Q}}^m[M, M^a]$ is finite (such that the integral of the total divergence in (23) is zero). We will see in Subsect. 5.1 that the coefficients of μ, j_a and σ_{ab} in the volume integral of (23) are precisely the various 3+1 parts of the Killing operator $\nabla^{(a}K^{b)}$ acting on $K^a := Mt^a + M^a$. Thus for Killing vectors $\mathtt{Q}^m[M, M^a]$ is constant in time even if the "time evolution" is defined by $\xi^a = Nt^a + N^a$ with asymptotically linearly diverging N and N^a given by (21)–(22).

The question of whether the evolution equations preserve the boundary conditions was investigated first by Beig and Ó Murchadha [5]. However, they considered only the vacuum equations with the $1/r$ and $1/r^2$ a priori fall-off of the metric and extrinsic curvature, respectively, and they assumed a priori that the lapse and the shift are time independent. While the first two are not serious limitations of their investigations, we do not see any reason to assume the time independence of N and N^a. In fact, the evolution equations allow their arbitrary time dependence, and, as we will see, the assumption of their time independence is too restrictive and we should abandon this.

Finally, for later convenience, it seems natural to introduce two notations here. We will denote by $\mathcal{A}$ the set of all the pairs (N, N^a) of lapses and shifts with the asymptotic form (21)–(22). Such pairs may be called the "allowed time axes", and obviously $\mathcal{A}$ can be endowed with a natural real vector space structure. We denote by $\mathcal{G}$ the subspace of $\mathcal{A}$ consisting of those pairs in which the 'parameters' $\beta_{\mathbf{k}}(t)$, $\rho_{\mathbf{ki}}(t)$, $\tau(t)$ and $\tau_{\mathbf{i}}(t)$ are all vanishing identically. We will see in the next subsection that, for $k \geq 1$, the generators of the gauge transformations in the phase space of vacuum general relativity are precisely the elements of $\mathcal{G}$. Thus we refer to $\mathcal{G}$ as to the space of the gauge generators even for $k > 0$.

4 The Hamiltonian Phase Space of Vacuum GR

4.1 The Phase Space and the General Beig–Ó Murchadha Hamiltonian

The configuration space $\mathcal{Q}$ for the asymptotically flat spacetimes is the set of the (negative definite) metrics on the 3-manifold Σ, a typical spacelike Cauchy surface in spacetime, satisfying the fall-off and parity conditions of Subsect. 3.1. Recalling that a curve in $\mathcal{Q}$ is a smooth 1-parameter family of metrics $q_{ab}(u)$ and the tangent vector of this curve at the point $q_{ab} := q_{ab}(0) \in \mathcal{Q}$ is defined to be the derivative $\delta q_{ab} := (\mathrm{d}q_{ab}(u)/\mathrm{d}u)|_{u=0}$, the tangent vector δq_{ab} satisfies the same boundary conditions as the metric q_{ab} itself does. The space of the tangent vectors at q_{ab} is denoted by $T_{q_{ab}}\mathcal{Q}$. Recall also that a 1-form at the point $q_{ab} \in \mathcal{Q}$ is a symmetric tensor density on Σ, which, at the same time, is a linear mapping $\tilde{p}^{ab} : T_{q_{ab}}\mathcal{Q} \to \mathbb{R}$ defined explicitly by $\langle \tilde{p}^{ab}, \delta q_{ab}\rangle := \int_\Sigma \tilde{p}^{ab}\delta q_{ab}\mathrm{d}^3x$. However, the requirement that its action on the tangent vectors be finite restricts its asymptotic structure. Indeed, if we write $\tilde{p}^{ab} = \frac{1}{r^m}\tilde{p}^{(m)ab} + o(r^{-m})$ for some $m > 0$, then from $\langle \tilde{p}^{ab}, \delta q_{ab}\rangle < \infty$ we obtain that $m \geq 3 - k$, and if the equality holds in this inequality then the components $\tilde{p}^{(m)\mathbf{ij}}$ of the leading term must be odd parity functions of $\frac{x^{\mathbf{k}}}{r}$. The space of these 1-forms at q_{ab}, the cotangent space of $\mathcal{Q}$ at q_{ab}, is denoted by $T^*_{q_{ab}}\mathcal{Q}$.

The phase space of vacuum general relativity is the cotangent bundle $T^*\mathcal{Q} := \{(\tilde{p}^{ab}, q_{ab})| +$ boundary conditions$\}$ of the configuration space with its natural symplectic structure: If $\mathcal{X} := (\delta\tilde{p}^{ab}, \delta q_{ab})$ and $\mathcal{X}' := (\delta'\tilde{p}^{ab}, \delta' q_{ab})$ are any two tangent vectors at some point $(\tilde{p}^{ab}, q_{ab}) \in T^*\mathcal{Q}$, then let $2\Omega_{(\tilde{p}^{ab},q_{ab})}(\mathcal{X},\mathcal{X}') := \int_\Sigma(\delta\tilde{p}^{ab}\delta' q_{ab} - \delta'\tilde{p}^{ab}\delta q_{ab})\mathrm{d}^3x$. Then the boundary conditions for the metrics and the canonical momenta ensure that $\Omega(\mathcal{X},\mathcal{X}')$ is already finite.

On the other hand, the canonical momentum $\tilde{p}^{ab}$ is well known to be the expression

$$\tilde{p}^{ab} = \frac{1}{2\kappa}\sqrt{|q|}\Big(\chi^{ab} - \chi q^{ab}\Big) = \frac{1}{r^{k+1}}\tilde{P}^{(k+1)ab} + o^\infty\big(r^{-k-1}\big) \tag{24}$$

of the metric and the extrinsic curvature, where we gave its asymptotic expansion too. Here the components of $\tilde{P}^{(k+1)ab}$ in the ${}_0q_{ab}$-Cartesian coordinates are odd parity functions of $\frac{x^{\mathbf{k}}}{r}$. Therefore, comparing this fall-off rate with the condition $m \geq 3-k$ obtained above, we find that *the applicability of the basic concepts of the symplectic framework already excludes the slow fall-off metrics*, i.e. $k \geq 1$ must be assumed. Thus the a priori fall-off $1/r$ considered by Beig and Ó Murchadha is the slowest possible in the symplectic framework.

Four of the vacuum Einstein equations, ${}^4G_{ab}t^at^b = 0$ and ${}^4G_{bc}P^b_at^c = 0$, play the role of constraints in the initial value as well as in the Hamiltonian formulation of the theory. In the phase space context they are represented by the vanishing of the so-called constraint functions

$$C\big[\nu,\nu^a\big] := \int_\Sigma\left(-\frac{1}{2\kappa}\left(R + \frac{4\kappa^2}{|q|}\left[\frac{1}{2}\tilde{p}^2 - \tilde{p}^{ab}\tilde{p}_{ab}\right]\right)\sqrt{|q|}\nu - 2\big(D_a\tilde{p}^{ab}\big)\nu_b\right)\mathrm{d}^3x\ , \tag{25}$$

parameterized by pairs (ν,ν^a) of functions and vector fields, which may be functions of the external time coordinate as well. A tedious but straightforward calculation shows that *the constraint functions are finite and functionally differentiable with respect to the canonical variables on the whole phase space and close to a Lie algebra if and only if* $(\nu,\nu^a) \in \mathcal{G}$. Since, via the symplectic 2-form, they generate gauge motions in the constraint surface $\Gamma \subset T^*\mathcal{Q}$, *$\mathcal{G}$ can be identified with the space of the infinitesimal gauge generators of Einstein's theory of the vacuum asymptotically flat spacetimes.*

The dynamics in the phase space is generated by the Hamiltonian, whose general form is the sum of a constraint function and the integral of an appropriately chosen total divergence. This total divergence should be chosen in such a way that the corresponding Hamilton equations be just the correct evolution equations (19) and (20) [4]. Beig and Ó Murchadha [5] showed that *the Hamiltonian*

$$\begin{aligned} H\big[M, M^a\big] := C\big[M, M^a\big] &+ \int_\Sigma 2D_a\big(\tilde{p}^{ab}M_b\big)\mathrm{d}^3x \\ -\frac{1}{2\kappa}\int_\Sigma D_a &\Big(Mq^{ab}q^{cd}\big({}_0D_c q_{bd} - {}_0D_b q_{cd}\big) \\ &+ \big({}_0D_b M\big)q^{ab}q^{cd}\big(q_{cd} - {}_0q_{cd}\big) \\ &- \big({}_0D_c M\big)q^{ab}q^{cd}\big(q_{bd} - {}_0q_{bd}\big)\Big)\sqrt{|q|}\mathrm{d}^3x \end{aligned} \tag{26}$$

is finite and functional differentiable with respect to the canonical variables on the whole phase space and close to a Lie algebra if and only if $(M, M^a) \in \mathcal{A}$. Thus we call H given by (26) the Beig–Ó Murchadha Hamiltonian. Note that M and M^a need not be time independent, they may still have arbitrary time dependence. The Poisson bracket of two Beig–Ó Murchadha Hamiltonians, parameterized by (M, M^a) and $(\bar{M}, \bar{M}^a)$, respectively, is

$$\begin{aligned} &\Big\{H\big[M, M^a\big], H\big[\bar{M}, \bar{M}^a\big]\Big\} \\ &\quad = -H\Big[\text{Ł}_{\mathbf{M}}\bar{M} - \text{Ł}_{\bar{\mathbf{M}}}M\ ,\ \big[M, \bar{M}\big]^a - \big(MD^a\bar{M} - \bar{M}D^aM\big)\Big]\ . \end{aligned} \tag{27}$$

Furthermore, for infinitesimal gauge generators the Hamiltonian of Beig and Ó Murchadha reduces to a constraint function: $H[\nu, \nu^a] = C[\nu, \nu^a]$. Therefore, *the Beig–Ó Murchadha Hamiltonians, parameterized by the elements of $\mathcal{A}$, form a Poisson algebra $\mathcal{H}$, in which the constraints, parameterized by the elements of $\mathcal{G}$, form an ideal $\mathcal{C}$.* The quotient $\mathcal{H}/\mathcal{C}$, which is again a Lie algebra, is the set of the Hamiltonians modulo the "gauge transformations". However, this quotient Lie algebra is spanned by the time dependent parameters $\beta_{\mathbf{k}}(t)$, $\rho_{\mathbf{ki}}(t)$, $\tau(t)$ and $\tau_{\mathbf{i}}(t)$, and hence it is infinite dimensional.

4.2 Physical Quantities from the Beig–Ó Murchadha Hamiltonians with Time-Independent Lapses and Shifts

As we mentioned, Beig and Ó Murchadha concentrated on the $k = 1$ case and assumed that M and M^a were time independent:

$$M\big(x^{\mathbf{k}}\big) = 2x^{\mathbf{k}}B_{\mathbf{k}} + T + \nu^{(0)}\Big(\frac{x^{\mathbf{k}}}{r}\Big) + o^\infty\big(r^{-0}\big)\,, \tag{28}$$

$$M_{\mathbf{i}}\big(x^{\mathbf{k}}\big) = 2x^{\mathbf{k}}R_{\mathbf{ki}} + T_{\mathbf{i}} + \nu^{(0)}_{\mathbf{i}}\Big(\frac{x^{\mathbf{k}}}{r}\Big) + o^\infty\big(r^{-0}\big)\ . \tag{29}$$

Here $B_{\mathbf{k}}$, $R_{\mathbf{ki}}$, T and $T_{\mathbf{i}}$ are real constants. The space of such pairs (M, M^a) will be denoted by ${}_0\mathcal{A}$, and the subspace of the "infinitesimal time independent gauge generators" by ${}_0\mathcal{G}$. Then the Beig–Ó Murchadha Hamiltonians parameterized by the elements of ${}_0\mathcal{A}$ form a Poisson algebra ${}_0\mathcal{H}$, in which the constraints parameterized by the elements of ${}_0\mathcal{G}$ form a Lie ideal ${}_0\mathcal{C}$. The quotient Lie algebra ${}_0\mathcal{H}/{}_0\mathcal{C}$ is spanned by the ten real parameters $B_{\mathbf{k}}$, $R_{\mathbf{ki}}$, T and $T_{\mathbf{i}}$, and, as Beig and Ó Murchadha showed, this is isomorphic to the Poincaré Lie algebra.

Since $H[M, M^a]$ is linear in M and M^a, its restriction to the constraint surface Γ is a 2-surface integral at infinity of the boundary expression in (26), which involves the parameters T, $T_{\mathbf{i}}$, $R_{\mathbf{ki}}$ and $B_{\mathbf{k}}$ linearly. The coefficients of these parameters define the total energy, linear momentum, spatial angular momentum and centre-of-mass, respectively:

$$H\big[M, M^a\big]|_{\Gamma} =: T\mathtt{P}^{\mathbf{0}} + T_{\mathbf{i}}\mathtt{P}^{\mathbf{i}} + R_{\mathbf{ij}}\mathtt{J}^{\mathbf{ij}} + 2B_{\mathbf{i}}\mathtt{J}^{\mathbf{i0}} \,. \tag{30}$$

The total energy and linear momentum defined in this way is precisely the familiar ADM energy and linear momentum [3], and the spatial angular momentum is just the angular momentum of Regge and Teitelboim [4]. However, the centre-of-mass expression deviates slightly from that given by Regge and Teitelboim. While the Regge–Teitelboim centre-of-mass is not always finite, the expression given by the Beig–Ó Murchadha Hamiltonian is. We call the latter expression the Beig–Ó Murchadha centre-of-mass.

4.3 Transformation and Conservation Properties

We saw in Subsect. 2.2 that even the quasi-locally defined energy-momentum and (relativistic) angular momentum of the matter fields transform in the correct way under the Poincaré transformations of the Cartesian coordinates. Since these transformations can also be interpreted as the action of the symmetries of the Minkowski spacetime, it is natural to ask about the transformation properties of the total energy, linear momentum, spatial angular momentum and centre-of-mass, introduced in the previous subsection, under the action of the "asymptotic symmetries" of the spacetime. Roughly, the structure of (28) and (29) is similar to the structure of the time and space projections of the Killing fields (1), thus it seems natural to identify them as the "asymptotic symmetry generators". Hence we would have to define the action of them on the physical quantities in question.

However, since $\mathtt{P}^{\mathbf{0}}$, $\mathtt{P}^{\mathbf{i}}$, $\mathtt{J}^{\mathbf{ij}}$ and $\mathtt{J}^{\mathbf{i0}}$ were introduced in the phase space rather than the spacetime, one may think that it is enough to clarify their transformation properties in the *phase space*. To do this we need an implementation of the "asymptotic symmetry generators" in the phase space in the form of some functionally differentiable function. However, we already do have such an implementation, namely the Beig–Ó Murchadha Hamiltonian parameterized by the 'symmetry generators', and hence we can define its action. The action of the "symmetry generator" $(\bar{M}, \bar{M}^a) \in {}_0\mathcal{A}$ on the total energy, linear momentum, spatial angular momentum and centre-of-mass is defined by the value on the constraint surface Γ of the Poisson bracket of the Hamiltonian implementing the 'asymptotic symmetry' and the Hamiltonian defining the physical quantities. Formally it is

$$\delta_{(\bar{M},\bar{M}^a)}\Big(T\mathtt{P}^{\mathbf{0}}+T_{\mathbf{i}}\mathtt{P}^{\mathbf{i}}+R_{\mathbf{ij}}\mathtt{J}^{\mathbf{ij}}+2B_{\mathbf{i}}\mathtt{J}^{\mathbf{i0}}\Big) := \Big\{H\big[\bar{M},\bar{M}^a\big],H\big[M,M^a\big]\Big\}|_{\Gamma} \ . \quad (31)$$

Evaluating the right hand side of (31) by using (27), the result can be summarized as follows: If we form the column vectors

$$\mathtt{P}^{\underline{a}} := \begin{pmatrix}\mathtt{P}^{\mathbf{0}}\\ \mathtt{P}^{\mathbf{i}}\end{pmatrix}, \qquad \bar{c}^{\underline{a}} := \begin{pmatrix}\bar{T}^{\mathbf{0}}\\ \bar{T}^{\mathbf{i}}\end{pmatrix},$$

and the 4×4 anti-symmetric matrices

$$\mathtt{J}^{\underline{a}\,\underline{b}} := \begin{pmatrix}0 & -\mathtt{J}^{\mathbf{i0}}\\ \mathtt{J}^{\mathbf{i0}} & \mathtt{J}^{\mathbf{ij}}\end{pmatrix}, \qquad \bar{\lambda}_{\underline{a}\,\underline{b}} := \begin{pmatrix}0 & -2\bar{B}_{\mathbf{j}}\\ 2\bar{B}_{\mathbf{i}} & 2\bar{R}_{\mathbf{ij}}\end{pmatrix},$$

then we obtain

$$\delta_{(\bar{M},\bar{M}^e)}\mathtt{P}^{\underline{a}} = -\mathtt{P}^{\underline{b}}\,\bar{\lambda}_{\underline{b}}{}^{\underline{a}} \quad (32)$$

$$\delta_{(\bar{M},\bar{M}^e)}\mathtt{J}^{\underline{a}\,\underline{b}} = -\Big(\mathtt{J}^{\underline{c}\,\underline{b}}\,\bar{\lambda}_{\underline{c}}{}^{\underline{a}} + \mathtt{J}^{\underline{a}\,\underline{c}}\,\bar{\lambda}_{\underline{c}}{}^{\underline{b}} + \big(\bar{c}^{\underline{a}}\,\mathtt{P}^{\underline{b}} - \bar{c}^{\underline{b}}\,\mathtt{P}^{\underline{a}}\big)\Big) \ . \quad (33)$$

This is precisely (minus) the action of the infinitesimal Poincaré transformation, parameterized by $\bar{c}^{\underline{a}} \in \mathbb{R}^4$ and the Lorentz Lie algebra element $\bar{\lambda}_{\underline{b}}{}^{\underline{a}}$, on an energy-momentum 4-vector and a relativistic angular momentum 4-tensor. Therefore, the total energy, linear momentum, spatial angular momentum and the Beig–Ó Murchadha centre-of-mass form Lorentz-covariant quantities, and transform *in the phase space* in the correct way.

The next issue that we should discuss is whether these quantities are conserved in time, or, more generally, under what conditions on the lapse and shift defining the time evolution do we have conserved total energy-momentum and (relativistic) angular momentum. Thus let $(N,N^a)\in\mathcal{A}$ be any allowed (maybe time dependent) time axis, given explicitly by (21) and (22) with $k=1$. Then we define the time derivative of $\mathtt{P}^{\underline{a}}$ and $\mathtt{J}^{\underline{a}\,\underline{b}}$ by the value on the constraint surface of the Poisson bracket of the Hamiltonian defining the time evolution via the dynamical equations and the Hamiltonian defining the physical quantities:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(T_{\underline{a}}\,\mathtt{P}^{\underline{a}} + M_{\underline{a}\,\underline{b}}\,\mathtt{J}^{\underline{a}\,\underline{b}}\Big) := \Big\{H\big[N,N^a\big],H\big[M,M^a\big]\Big\}\Big|_{\Gamma} \ . \quad (34)$$

Evaluating the right hand side of (34) by using (27), for the time independence of the physical quantities above we obtain the following list:

$$\dot{\mathtt{P}}^{\mathbf{0}} = 0 \quad \text{iff} \quad \beta_{\mathbf{k}}(t) = 0\,, \quad (35)$$

$$\dot{\mathtt{P}}^{\mathbf{i}} = 0 \quad \text{iff} \quad \beta_{\mathbf{k}}(t) = 0\,,\ \rho_{\mathbf{ki}}(t) = 0\,, \quad (36)$$

$$\dot{\mathtt{J}}^{\mathbf{ij}} = 0 \quad \text{iff} \quad \beta_{\mathbf{k}}(t) = 0\,,\ \rho_{\mathbf{ki}}(t) = 0\,,\ \tau_{\mathbf{i}}(t) = 0\,, \quad (37)$$

$$\dot{\mathtt{J}}^{\mathbf{i0}} = 0 \quad \text{iff} \quad \beta_{\mathbf{k}}(t) = 0\,,\ \rho_{\mathbf{ki}}(t) = 0,,\ \tau_{\mathbf{i}}(t) = 0,\ \tau(t) = 0\,. \quad (38)$$

Therefore, *the total ADM energy-momentum* $\mathtt{P}^{\underline{a}}$ *and the relativistic angular momentum* $\mathtt{J}^{\underline{a}\,\underline{b}}$*, built from the spatial Regge–Teitelboim angular momentum and the Beig–Ó Murchadha centre-of-mass, are conserved only with respect to gauge evolutions, i.e. when* $(N, N^a) \in \mathcal{G}$.

4.4 Three Difficulties

In Subsects. 2.5 and 3.2 we found that the lapse and the shift that ensure the conservation of the total energy-momentum and (relativistic) angular momentum of the matter fields may even be asymptotically linearly diverging, i.e. they may be any element of $\mathcal{A}$. In the light of this result it is quite surprising that the analogous gravitational quantities are conserved only with respect to considerably more restricted lapses and shifts: These must tend to zero at infinity, and, in particular, the Beig–Ó Murchadha centre-of-mass is not conserved even with respect to time evolution that is a pure asymptotic time translation at infinity. Thus we raise the question of whether the total energy-momentum and (relativistic) angular momentum introduced above are really the "ultimate" expressions, or whether there is a slightly different definition for them with better conservation properties. We expect that these total quantities must be conserved at least with respect to pure asymptotic time translations.

However, there is a second difficulty too. Although we noted in Subsects. 3.2 and 4.3 that the structure of the allowed lapses and shifts are only *roughly* similar to that of the time and space projections of the Killing fields in Minkowski spacetime, respectively, in Subsect. 4.3 we swept this observation under the rug, and we considered the elements of ${}_0\mathcal{A}$ as the lapse and shift parts of the generators of the 'asymptotic symmetries' of the spacetime. Nevertheless, strictly speaking, neither the elements of $\mathcal{A}$ nor of ${}_0\mathcal{A}$ can be identified with the generators of the asymptotic symmetries of the spacetime. Indeed, while the elements of $\mathcal{A}$ have arbitrary time dependence and the elements of ${}_0\mathcal{A}$ are completely time independent, the components of the Killing vectors of the Minkowski spacetime have a very specific, namely *linear* time dependence. In particular, the familiar boost Killing vectors of the Minkowski spacetime cannot be recovered, neither from $\mathcal{A}$ nor from ${}_0\mathcal{A}$, in the weak field approximation.

The third difficulty is that while the centre-of-mass of the matter fields in Minkowski spacetime depends on the Cartesian time coordinate, the Beig–Ó Murchadha centre-of-mass is completely time independent. But the time dependence of the centre-of-mass was needed to prove not only its conservation, but also its correct transformation properties in the spacetime. Although the relativistic angular momentum built from the spatial angular momentum and the Beig–Ó Murchadha centre-of-mass transforms in the correct way *in the phase space*, this does not imply its correct transformation *in the spacetime.*

In the rest of this contribution we try to resolve these three problems by showing first how the "correct" time dependence of the lapse functions can be obtained. Since these resolutions grew up from the need to have a systematic spacetime interpretation of the results and the analysis of Beig and Ó Murchadha, we go back to spacetime.

5 The Asymptotic Spacetime Killing Vectors

5.1 The 3 + 1 Form of the Lie Brackets and the Killing Operators

Let Σ be a smooth spacelike hypersurface with future pointing timelike unit normal t^a and induced metric q_{ab}. Let K^a and $\bar{K}^a$ be two arbitrary vector fields on M, and let their $3+1$ decomposition on Σ be $K^a = Mt^a + M^a$ and $\bar{K}^a = \bar{M}t^a + \bar{M}^a$. Then the $3+1$ decomposition of their Lie bracket with respect to Σ can be written as

$$\begin{aligned}\left[K,\bar{K}\right]^a &= \left(t^a t^b + 2q^{ab}\right)\Big(M\nabla_{(b}\bar{K}_{c)} - \bar{M}\nabla_{(b}K_{c)}\Big)t^c \\ &\quad + t^a\left(\text{Ł}_{\mathbf{M}}\bar{M} - \text{Ł}_{\bar{\mathbf{M}}}M\right) + \Big(\left[M,\bar{M}\right]^a - \left(MD^a\bar{M} - \bar{M}D^aM\right)\Big)\,. \qquad (39)\end{aligned}$$

Observe that the first two terms on the right are the time-time and the time-space projections of the spacetime Killing operators acting on K^a and $\bar{K}^a$. The third term on the right is precisely the combination of the lapse and shift parts of K^a and $\bar{K}^a$ that appeared as the new lapse in the calculation of the Poisson bracket of two Beig–Ó Murchadha Hamiltonians (27). Similarly, the last term is built from M, M^a, $\bar{M}$ and $\bar{M}^a$ precisely in the same way as the new shift from the old lapses and shifts in (27). Thus one can expect that the Lie bracket of spacetime vector fields plays some role in the Poisson algebra of the Beig–Ó Murchadha Hamiltonians. Parts of the Killing operator are vanishing in some sense. Therefore it is worth decomposing the Killing operator in the $3+1$ way as well.

Although the space-space projection of the Killing operator can be expressed by three dimensional quantities defined with respect to Σ, the time-time and the time-space projections can be done only if we have not only a single spacelike hypersurface, but a whole foliation and a notion of "time flow" ξ^a as well. Thus we fix the vector field ξ^a, which will be represented by a lapse and a shift according to $\xi^a = Nt^a + N^a$. If $\dot{X}^a$ denotes the projection of the Lie derivative of the *spatial* X^a along ξ^a, then the full $3+1$ decomposition of $\nabla^{(a}K^{b)}$ is

$$Nt_c t_d \nabla^{(c}K^{d)} = \dot{M} + \text{Ł}_{\mathbf{M}}N - \text{Ł}_{\mathbf{N}}M\,, \qquad (40)$$

$$2NP^a_c t_d \nabla^{(c}K^{d)} = \dot{M}^a + \left(ND^aM - MD^aN\right) - \left[N,M\right]^a\,, \qquad (41)$$

$$P^a_c P^b_d \nabla^{(c}K^{d)} = D^{(a}M^{b)} + M\chi^{ab}\,. \qquad (42)$$

Recall that precisely these projections appeared in (23). Furthermore, apart from the dot-derivatives, the right hand side of (40) and (41) are precisely the special combinations of the lapses and shifts that already appeared in (27). Equations (39–41) will be our key equations.

5.2 The Asymptotic Killing Vectors

In Subsect. 3.2 we introduced $\mathcal{A}$ as the space of the allowed, most general lapse-shift pairs compatible with the boundary conditions via the evolution equations. Thus in this picture $\mathcal{A}$ is the space of the allowed spacetime coordinate systems based on a single, fixed asymptotically flat spacelike hypersurface Σ. Two elements of $\mathcal{A}$, say (M, M^a) and (M', M'^a), determine two different foliations of the spacetime, and the corresponding unit timelike normals, t^a and t'^a, are different.

However, we can look at the space $\mathcal{A}$ from a slightly different perspective too. Let us fix a vector field ξ^a, which determines a foliation of the spacetime that is based on the single asymptotically flat Σ. Let t^a be the future pointing unit timelike normal of the leaves of this foliation, and let the lapse and the shift parts of ξ^a be chosen to be allowed: $(N, N^a) \in \mathcal{A}$, where $Nt^a + N^a = \xi^a$. Then for any $(M, M^a) \in \mathcal{A}$ define the spacetime vector field $K^a := Mt^a + M^a$. Note that we use the *same* t^a to define K^a for all (M, M^a). Thus the role of ξ^a is to provide a differential topological background to build spacetime vector fields from the pairs (M, M^a). The space of such spacetime vector fields will be denoted by $\mathcal{A}_\xi$, and let $\mathcal{G}_\xi$ be its subspace whose elements are constructed using $\mathcal{G}$ instead of $\mathcal{A}$.

Next observe that the space-space projection of the Killing operator (42) acting on any vector field $K^a \in \mathcal{A}_\xi$ is vanishing asymptotically at least as $O(r^{-k})$, and if this fall-off is actually $O(r^{-k})$ then the leading term has even parity. However, its time-time and time-space projections can still be arbitrary. This motivates us how to define the asymptotic Killing vectors: The vector field $K^a \in \mathcal{A}_\xi$ will be called an *asymptotic Killing vector with respect to* ξ^a if $t^c t^d \nabla_{(c} K_{d)}$ and $P^c_a t^d \nabla_{(c} K_{d)}$ are also vanishing asymptotically at least as $O(r^{-k})$, and if this fall-off is actually $O(r^{-k})$ then the leading terms have even parity. We can introduce a slightly stronger notion: $K^a \in \mathcal{A}_\xi$ will be called a *strongly asymptotic Killing vector with respect to* ξ^a if $t^c t^d \nabla_{(c} K_{d)} = 0$ and $P^c_a t^d \nabla_{(c} K_{d)} = 0$, i.e. when the right side of (40) and (41) is vanishing not only asymptotically, but pointwise as well. Note that although the Killing equation has only the trivial solution in a general spacetime, the asymptotic Killing and the strong asymptotic Killing equations can always be solved among the vector fields $K^a \in \mathcal{A}_\xi$.

Indeed, $t^c t^d \nabla_{(c} K_{d)} = O(r^{-k})$ and $P^c_a t^d \nabla_{(c} K_{d)} = O(r^{-k})$ are *not* partial differential equations, they are only ordinary differential equations for the time dependence of M and M^a. In particular, if the asymptotic structure of the lapse N and the shift N^a is given by (21) and (22), respectively, and the asymptotic structure of $(M, M^a) \in \mathcal{A}$ is

$$M\big(t,x^{\mathbf{k}}\big)=2x^{\mathbf{k}}B_{\mathbf{k}}\big(t\big)+T\big(t\big)+r^{G}\mu^{(G)}\Big(t,\frac{x^{\mathbf{k}}}{r}\Big)+o^{\infty}\big(r^{G}\big)\,, \tag{43}$$

$$M_{\mathbf{i}}\big(t,x^{\mathbf{k}}\big)=2x^{\mathbf{k}}R_{\mathbf{ki}}\big(t\big)+T_{\mathbf{i}}\big(t\big)+r^{H}\mu^{(H)}_{\mathbf{i}}\Big(t,\frac{x^{\mathbf{k}}}{r}\Big)+o^{\infty}\big(r^{H}\big)\,, \tag{44}$$

where $G,H\leq(1-k)$, then both the asymptotic and the strong asymptotic Killing equations give the ordinary differential equations

$$\dot{B}_{\mathbf{i}}=-2\Big(R_{\mathbf{ij}}\beta^{\mathbf{j}}-\rho_{\mathbf{ij}}B^{\mathbf{j}}\Big)\,, \tag{45}$$

$$\dot{R}_{\mathbf{ij}}=2\Big(B_{\mathbf{i}}\beta_{\mathbf{j}}-\beta_{\mathbf{i}}B_{\mathbf{j}}\Big)-2\Big(R_{\mathbf{ik}}\rho^{\mathbf{k}}{}_{\mathbf{j}}-\rho_{\mathbf{ik}}R^{\mathbf{k}}{}_{\mathbf{j}}\Big)\,, \tag{46}$$

and if $k\geq1$, we also have

$$\dot{T}=-2\Big(T_{\mathbf{i}}\beta^{\mathbf{i}}-\tau_{\mathbf{i}}B^{\mathbf{i}}\Big)\,, \tag{47}$$

$$\dot{T}_{\mathbf{i}}=2\Big(T\beta_{\mathbf{i}}-\tau B_{\mathbf{i}}\Big)-2\Big(T^{\mathbf{j}}\rho_{\mathbf{ji}}-\tau^{\mathbf{j}}R_{\mathbf{ji}}\Big)\,. \tag{48}$$

(45)–(48) is a system of ordinary differential equations for $B_{\mathbf{i}}(t)$, $R_{\mathbf{ij}}(t)$, $T_{\mathbf{i}}(t)$ and $T(t)$. For given $\beta_{\mathbf{i}}(t)$, $\rho_{\mathbf{ij}}(t)$, $\tau_{\mathbf{i}}(t)$ and $\tau(t)$ this can always be solved, and the solution depends on six, and if $k\geq1$ then on ten constants of integration. Here raising and lowering of the boldface Roman indices are defined by the spatial projection of the constant Minkowski metric: $\eta_{\mathbf{ij}}=-\delta_{\mathbf{ij}}$. Thus the role of the asymptotic Killing equations is that they restrict the unspecified time dependence of M and M^a. In particular,

- if $\rho_{\mathbf{ij}}=0$, $\beta_{\mathbf{i}}=0$, $\tau_{\mathbf{i}}=0$ and $\tau=0$, i.e. if ξ^a is a pure gauge generator $(N,N^a)\in\mathcal{G}$, then the spacetime coordinate system that ξ^a defines is asymptotically *collapsing*. Then the solution of (45)-(48) is that $B_{\mathbf{i}}$, $R_{\mathbf{ij}}$, $T_{\mathbf{i}}$ and T are all constant;
- if $\rho_{\mathbf{ij}}=0$, $\beta_{\mathbf{i}}=0$, $\tau_{\mathbf{i}}=0$ and $\tau=1$, i.e. if ξ^a is a pure asymptotic time translation, then the corresponding coordinate system is an *asymptotically Cartesian coordinate system*. Then $B_{\mathbf{i}}$, $R_{\mathbf{ij}}$ and T are constant but $T_{\mathbf{i}}(t)=T_{\mathbf{i}}-2tB_{\mathbf{i}}$ for some constants $T_{\mathbf{i}}$;
- if $\rho_{\mathbf{ij}}=0$, $\tau=0$, $\beta_{\mathbf{i}}=$ const. with $\beta_{\mathbf{i}}\beta_{\mathbf{j}}\delta^{\mathbf{ij}}=1$ and $\tau_{\mathbf{i}}(t)=-2t\beta_{\mathbf{i}}$, then the corresponding coordinates form an *asymptotically Rindler coordinate system*. Then the solution of (45)–(48) is considerably more complicated:

$$\begin{aligned}
B_{\mathbf{i}}(t) &= -\beta_{\mathbf{i}}\beta^{\mathbf{k}}B_{\mathbf{k}} + \Pi_{\mathbf{i}}^{\mathbf{k}}B_{\mathbf{k}}\cosh(2t) - R_{\mathbf{ik}}\beta^{\mathbf{k}}\sinh(2t)\,,\\
R_{\mathbf{ij}}(t) &= \Pi_{\mathbf{i}}^{\mathbf{k}}\Pi_{\mathbf{j}}^{\mathbf{l}}R_{\mathbf{kl}}\\
&\quad + \big(\beta_{\mathbf{i}}R_{\mathbf{jk}} - \beta_{\mathbf{j}}R_{\mathbf{ik}}\big)\beta^{\mathbf{k}}\cosh(2t) - \big(\beta_{\mathbf{i}}\Pi_{\mathbf{j}}^{\mathbf{k}} - \beta_{\mathbf{j}}\Pi_{\mathbf{i}}^{\mathbf{k}}\big)B_{\mathbf{k}}\sinh(2t)\,,\\
T(t) &= \beta^{\mathbf{k}}B_{\mathbf{k}} + \big(T - \beta^{\mathbf{k}}B_{\mathbf{k}}\big)\cosh(2t) - \beta^{\mathbf{k}}T_{\mathbf{k}}\sinh(2t)\,,\\
T_{\mathbf{i}}(t) &= \Pi_{\mathbf{i}}^{\mathbf{k}}T_{\mathbf{k}} + R_{\mathbf{ik}}\beta^{\mathbf{k}} + 2t\beta_{\mathbf{i}}\beta^{\mathbf{k}}B_{\mathbf{k}}\\
&\quad - \Big(\beta_{\mathbf{i}}\beta^{\mathbf{k}}T_{\mathbf{k}}2t\Pi_{\mathbf{i}}^{\mathbf{k}}B_{\mathbf{k}} + R_{\mathbf{ik}}\beta^{\mathbf{k}}\Big)\cosh(2t)\\
&\quad + \Big(\beta_{\mathbf{i}}\big(T - \beta^{\mathbf{k}}B_{\mathbf{k}}\big) + \Pi_{\mathbf{i}}^{\mathbf{k}}B_{\mathbf{k}} + 2tR_{\mathbf{ik}}\beta^{\mathbf{k}}\Big)\sinh(2t)\;,
\end{aligned}$$

where $\Pi_{\mathbf{i}}^{\mathbf{k}} := \delta_{\mathbf{i}}^{\mathbf{k}} + \beta_{\mathbf{i}}\beta^{\mathbf{k}}$ is the projection to the 2-plane orthogonal to $\beta_{\mathbf{i}}$ and T, $T_{\mathbf{i}}$, $B_{\mathbf{i}}$ and $R_{\mathbf{ij}}$ are constants.

Therefore, both the time independent generators of Beig and Ó Murchadha and the familiar Killing vectors of the Minkowski spacetime can be recovered as asymptotic Killing vectors by an appropriate choice for ξ^a, and the latter is connected with the asymptotically Cartesian coordinate system discussed in Subsect. 2.4.

The space of the asymptotic Killing vectors and of the strong asymptotic Killing vectors (with respect to ξ^a) will be denoted by $\mathcal{A}^K_\xi$ and $\mathcal{A}^0_\xi$, respectively, and obviously $\mathcal{A}^0_\xi \subset \mathcal{A}^K_\xi \subset \mathcal{A}_\xi$.

5.3 The Algebra of Asymptotic Symmetries

Contrary to expectations, the space $\mathcal{A}_\xi$ does not close to a Lie algebra with respect to the spacetime Lie bracket. To see this, it is enough the consider the t^a component of the Lie bracket given by (39) and take into account that $\text{Ł}_{\mathbf{M}}\bar{M} - \text{Ł}_{\bar{\mathbf{M}}}M$ has the form of an allowed lapse for any $(M, M^a), (\bar{M}, \bar{M}^a) \in \mathcal{A}$, while the leading term in $t^at^b(M\nabla_{(a}\bar{K}_{b)} - \bar{M}\nabla_{(a}K_{b)})$ has the form $N^{-1}x^{\mathbf{i}}x^{\mathbf{j}}$, which deviates from the structure of the allowed lapses.

If K^a and $\bar{K}^a$ are any two asymptotic Killing vectors then by (39) their Lie bracket, $\hat{K}^a := [K, \bar{K}]^a$, belongs to $\mathcal{A}_\xi$. Furthermore, the (time dependent) parameters in its asymptotic expansion according to (43) and (44), $\hat{B}_{\mathbf{i}}$ and $\hat{R}_{\mathbf{ij}}$, and if $k \geq 1$ then $\hat{T}_{\mathbf{i}}$ and $\hat{T}$ also, are built from those of K^a and $\bar{K}^a$ as

$$\hat{B}_{\mathbf{i}} = 2\big(R_{\mathbf{ij}}\bar{B}^{\mathbf{j}} - \bar{R}_{\mathbf{ij}}B^{\mathbf{j}}\big)\,, \tag{49}$$

$$\hat{R}_{\mathbf{ij}} = 2\big(R_{\mathbf{ik}}\bar{R}^{\mathbf{k}}{}_{\mathbf{j}} - \bar{R}_{\mathbf{ik}}R^{\mathbf{k}}{}_{\mathbf{j}} + \bar{B}_{\mathbf{i}}B_{\mathbf{j}} - B_{\mathbf{i}}\bar{B}_{\mathbf{j}}\big)\,, \tag{50}$$

$$\hat{T}_{\mathbf{i}} = 2\big(T^{\mathbf{j}}\bar{R}_{\mathbf{ji}} - \bar{T}^{\mathbf{j}}R_{\mathbf{ji}} + \bar{T}B_{\mathbf{i}} - T\bar{B}_{\mathbf{i}}\big)\,, \tag{51}$$

$$\hat{T} = 2\big(T_{\mathbf{i}}\bar{B}^{\mathbf{i}} - \bar{T}_{\mathbf{i}}B^{\mathbf{i}}\big)\;. \tag{52}$$

Now it is a direct calculation to show that $\hat{B}_{\mathbf{i}}$, $\hat{R}_{\mathbf{ij}}$, $\hat{T}_{\mathbf{i}}$ and $\hat{T}$ satisfy (45)-(48). Thus the leading, and if $k \geq 1$ then the leading two terms in $[K, \bar{K}]^a$

satisfy even the strong asymptotic Killing equations. However, in general $[K, \bar{K}]^a$ does not satisfy the asymptotic Killing equations even if both K^a and $\bar{K}^a$ are strong asymptotic Killing vectors. To see this we should calculate the projections $P^c_a P^d_b \text{Ł}_{[\mathbf{K},\bar{\mathbf{K}}]} g_{cd}$, $P^c_a t^d \text{Ł}_{[\mathbf{K},\bar{\mathbf{K}}]} g_{cd}$ and $t^c t^d \text{Ł}_{[\mathbf{K},\bar{\mathbf{K}}]} g_{cd}$. Using the differential geometric identity $L_{[\mathbf{K},\bar{\mathbf{K}}]} = \text{Ł}_{\mathbf{K}}\text{Ł}_{\bar{\mathbf{K}}} - \text{Ł}_{\bar{\mathbf{K}}}\text{Ł}_{\mathbf{K}}$, it is a straightforward calculation to show that $P^c_a t^d \text{Ł}_{[\mathbf{K},\bar{\mathbf{K}}]} g_{cd}$ is not of order $O(r^{-k})$ for general $K^a, \bar{K}^a \in \mathcal{A}^K_\xi$, and it is not zero for general $K^a, \bar{K}^a \in \mathcal{A}^0_\xi$. Therefore, neither $\mathcal{A}^K_\xi$ nor $\mathcal{A}^0_\xi$ close to a Lie algebra. Nevertheless, by the fact that $\hat{B}_{\mathbf{i}}$, $\hat{R}_{\mathbf{ij}}$, $\hat{T}_{\mathbf{i}}$ and $\hat{T}$ satisfy (45)–(48) the Lie bracket of any two asymptotic Killing vectors deviates from an asymptotic Killing field only by an element of $\mathcal{G}_\xi$. This observation makes it possible to introduce a natural Lie algebra structure on the quotient vector spaces $\mathcal{A}^K_\xi/\mathcal{G}^K_\xi$ and $\mathcal{A}^0_\xi/\mathcal{G}^0_\xi$, where $\mathcal{G}^K_\xi := \mathcal{G}_\xi \cap \mathcal{A}^K_\xi$ and $\mathcal{G}^0_\xi := \mathcal{G}_\xi \cap \mathcal{A}^0_\xi$. These quotient spaces are spanned by the (special time dependent) parameters $B_{\mathbf{i}}$ and $R_{\mathbf{ij}}$, and if $k \geq 1$ then also by $T_{\mathbf{i}}$ and T. Hence they are isomorphic to each other and their dimension is six for $k < 1$ and ten for $k \geq 1$. The Lie multiplication of them is given by (49)–(52), and it is easy to see that this Lie algebra is the Lorentz Lie algebra for $k < 1$ and the Poincaré algebra for $k \geq 1$. Therefore, *the structure of the Lie algebra $\mathcal{A}^K_\xi/\mathcal{G}^K_\xi$ is connected with the fall-off rate of the metric: for slow fall-off it is only the Lorentz Lie algebra, and the displacements of the origin of the coordinate system emerge as asymptotic symmetries only for $1/r$ or faster fall-off.*

6 Beig–Ó Murchadha Hamiltonians with Asymptotic Spacetime Killing Vectors

In this section we return to the discussion of the properties of the Beig–Ó Murchadha Hamiltonian, but instead of the elements of the time independent $(M, M^a) \in {}_0\mathcal{A}$ we parameterize them by the asymptotic Killing vectors.

Thus let us fix ξ^a, and define $H[K^a] := H[M, M^a]$ for any $K^a := Mt^a + M^a \in \mathcal{A}^K_\xi$. Then by (39) the Lie multiplication law (27) in the Poisson algebra of the Beig–Ó Murchadha Hamiltonians can be written in the remarkably simple form

$$\Big\{ H\big[K^a\big], H\big[\bar{K}^a\big] \Big\} = \begin{cases} -H\Big[\big[K, \bar{K}\big]^a\Big] + \text{constraints for } K^a, \bar{K}^a \in \mathcal{A}^K_\xi \,, \\ -H\Big[\big[K, \bar{K}\big]^a\Big] \qquad\qquad\qquad\quad \text{for } K^a, \bar{K}^a \in \mathcal{A}^0_\xi \,. \end{cases} \tag{53}$$

Therefore, *apart from constraints, the Beig–Ó Murchadha Hamiltonian preserves the spacetime Lie bracket of the asymptotic spacetime Killing vectors, and it preserves the spacetime Lie bracket of the strong asymptotic spacetime Killing vectors.*

The second issue that we consider is the conservation of the Hamiltonian. Thus let $(M, M^a) \in \mathcal{A}$, and calculate the *total* time derivative of $H[M, M^a]$, where the time evolution is generated by $\xi^a = Nt^a + N^a$. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} H\left[M, M^a\right] &= H\big[\dot{M}, \dot{M}^a\big] + \Big\{H\big[N, N^a\big], H\big[M, M^a\big]\Big\} \\ &= H\big[\dot{M} + M^e D_e N - N^e D_e M, \dot{M}^a + N D^a M - M D^a N - [N, M]^a\big] \\ &= \begin{cases} \text{constraints for } Mt^a + M^a \in \mathcal{A}^K_\xi\,, \\ 0 \qquad\qquad\quad \text{for } Mt^a + M^a \in \mathcal{A}^0_\xi\,. \end{cases}\end{aligned} \tag{54}$$

Here we used (27), and, in the last step, the definition of the asymptotic Killing and the strong asymptotic Killing vectors. Thus, *the Beig–Ó Murchadha Hamiltonian is constant (constant modulo constraints) with respect to the time evolution defined by* ξ^a *if* $K^a = Mt^a + M^a$ *is strongly asymptotic Killing (asymptotic Killing) with respect to* ξ^a.

7 Physical Quantities from the Beig–Ó Murchadha Hamiltonians with Asymptotic Spacetime Killing Vectors

7.1 The General Definition of the Physical Quantities

Independently of the details of the canonical analysis of the vacuum Einstein theory, we can consider the Beig–Ó Murchadha Hamiltonian as a functional of the initial data on an asymptotically flat spacelike hypersurface even in the presence of matter fields and even if the fall-off rate of the metric is assumed only to be positive. Thus for any $(M, M^a) \in \mathcal{A}$ let us define

$$\begin{aligned}\mathtt{Q}\big[M, M^a\big] &:= H\big[M, M^a\big]|_\Gamma + \mathtt{Q}^m\big[M, M^a\big] \\ &= -\frac{1}{2\kappa}\int_\Sigma D_a \ \Big(\ M q^{ab} q^{cd}\big({}_0D_c q_{bd} - {}_0D_b q_{cd}\big) \\ &\qquad + \big({}_0D_b M\big) q^{ab} q^{cd}\big(q_{cd} - {}_0q_{cd}\big) \\ &\qquad - \big({}_0D_c M\big) q^{ab} q^{cd}\big(q_{bd} - {}_0q_{bd}\big) \\ &\qquad - 2M_b\big(\chi^{ba} - \chi q^{ba}\big)\Big)\sqrt{|q|}\mathrm{d}^3x\,.\end{aligned} \tag{55}$$

Apparently, for zero $B_{\mathbf{i}}$ and $R_{\mathbf{ij}}$ but non-zero T or $T_{\mathbf{i}}$ this expression is finite only if $k \geq 1$. However, as it was pointed out in [6], [7], [8] in the vacuum case, $\mathtt{Q}[M, M^a]$ is finite even if $k > 1/2$ *and the fall-off rate* G *and* H *in (43)–(44) satisfies the stronger restriction* $G, H \leq -k$: Relaxing the fall off for the matter fields analogously, the right hand side can be written as the sum of a finite and a would-be divergent term, but the latter in fact vanishes by the constraint parts of the field equations. (Of course, in this case the energy-momentum of the matter fields is *not* finite.) Similarly, apparently $\mathtt{Q}[M, M^a]$ can be finite for non-zero $B_{\mathbf{i}}$ and $R_{\mathbf{ij}}$ only for $k \geq 2$, but, as an analogous analysis shows [14], the slowest possible fall-off rate ensuring the finiteness of (55) is in fact $k \geq 1$.

7.2 Total Energy, Momentum, Angular Momentum and Centre-of-Mass

Next let us restrict $K^a := Mt^a + M^a$ to be an asymptotic Killing vector and introduce the notation $\mathtt{Q}[K^a] := \mathtt{Q}[M, M^a]$. Then since $\mathcal{A}^K_\xi / \mathcal{G}^K_\xi \approx \mathcal{A}^0_\xi / \mathcal{G}^0_\xi$ is coordinatized by the integration *constants* $B_{\mathbf{i}}$ and $R_{\mathbf{ij}}$, and for $k \geq 1$ by $T_{\mathbf{i}}$ and T too, $\mathtt{Q}[K^a]$ is a linear expression of them:

$$\mathtt{Q}[K^a] = T\mathtt{P}^{\mathbf{0}} + T_{\mathbf{i}}\mathtt{P}^{\mathbf{i}} + R_{\mathbf{ij}}\mathtt{J}^{\mathbf{ij}} + 2B_{\mathbf{i}}\mathtt{J}^{\mathbf{i0}}. \tag{56}$$

This defines the total energy, linear momentum, spatial angular momentum and centre-of-mass, respectively. However, these quantities depend on the choice of the vector field ξ^a. In particular,

- if ξ^a is chosen to be a pure gauge generator, then we recover the ADM energy $\mathtt{P}^{\mathbf{0}}_{ADM}$, the ADM linear momentum $\mathtt{P}^{\mathbf{i}}_{ADM}$, the Regge–Teitelboim spatial angular momentum $\mathtt{J}^{\mathbf{ij}}_{RT}$ and the Beig–Ó Murchadha centre-of-mass $\mathtt{J}^{\mathbf{i0}}_{BOM}$, respectively;
- if ξ^a is a pure asymptotic time translation, then the energy, linear momentum and spatial angular momentum coincide with the ADM energy and linear momentum and the Regge–Teitelboim angular momentum, but the centre-of-mass deviates slightly from the Beig–Ó Murchadha centre-of-mass; it is $\mathtt{J}^{\mathbf{i0}} = \mathtt{J}^{\mathbf{i0}}_{BOM} - t\mathtt{P}^{\mathbf{i}}_{ADM}$;
- if ξ^a defines an asymptotically Rindler coordinate system, then the energy, linear momentum, spatial angular momentum and centre-of-mass will be complicated time dependent combinations of the ADM energy and linear momentum, the Regge–Teitelboim angular momentum and the Beig–Ó Murchadha centre-of-mass:

$$\begin{aligned}
\mathtt{P}^{\mathbf{0}} &= \mathtt{P}^{\mathbf{0}}_{ADM}\cosh(2t) + \beta_{\mathbf{k}}\mathtt{P}^{\mathbf{k}}_{ADM}\sinh(2t), \\
\mathtt{P}^{\mathbf{i}} &= \Pi^{\mathbf{i}}_{\mathbf{k}}\mathtt{P}^{\mathbf{k}}_{ADM} - \beta^{\mathbf{i}}\big(\mathtt{P}^{\mathbf{0}}_{ADM}\sinh(2t) + \beta_{\mathbf{k}}\mathtt{P}^{\mathbf{k}}_{ADM}\cosh(2t)\big)\,, \\
\mathtt{J}^{\mathbf{ij}} &= \Pi^{\mathbf{i}}_{\mathbf{k}}\Pi^{\mathbf{j}}_{\mathbf{l}}\mathtt{J}^{\mathbf{kl}}_{RT} + 2\beta^{[\mathbf{i}}\mathtt{J}^{\mathbf{j}]\mathbf{k}}_{RT}\beta_{\mathbf{k}}\cosh(2t) + 2\beta^{[\mathbf{i}}\mathtt{J}^{\mathbf{j}]\mathbf{0}}_{BOM}\sinh(2t) \\
&\quad - \beta^{[\mathbf{i}}\mathtt{P}^{\mathbf{j}]}_{ADM}\big(1 - \cosh(2t) + 2t\sinh(2t)\big)\,, \\
\mathtt{J}^{\mathbf{i0}} &= -\beta^{\mathbf{i}}\beta_{\mathbf{k}}\mathtt{J}^{\mathbf{k0}}_{BOM} + \Pi^{\mathbf{i}}_{\mathbf{k}}\big(\mathtt{J}^{\mathbf{k0}}_{BOM}\cosh(2t) + \mathtt{J}^{\mathbf{kl}}_{RT}\beta_{\mathbf{l}}\sinh(2t)\big) \\
&\quad + \frac{1}{2}\mathtt{P}^{\mathbf{i}}_{ADM}\sinh(2t) + t\big(\beta^{\mathbf{i}}\beta_{\mathbf{k}} - \Pi^{\mathbf{i}}_{\mathbf{k}}\cosh(2t)\big)\mathtt{P}^{\mathbf{i}}_{ADM} \\
&\quad + \frac{1}{2}\beta^{\mathbf{i}}\big(1 - \cosh(2t)\big)\mathtt{P}^{\mathbf{0}}_{ADM}\,.
\end{aligned}$$

Thus the definition of the physical quantities, defined by the value of the Beig–Ó Murchadha Hamiltonian parameterized by the asymptotic spacetime Killing vectors, do depend on the vector field ξ^a that we used to define the asymptotic Killing vectors. Hence we should have a selection rule for ξ^a. Based on the discussions in Subsect. 2.4, such a selection rule could be the requirement that the spacetime coordinate system determined by ξ^a be asymptotically Cartesian. Our suggestion is to take such a ξ^a. In fact, this choice should be justified by the properties of the corresponding physical quantities.

The analysis of Subsect. 4.3 to clarify the transformation properties of this total energy, linear momentum, spatial angular momentum and centre-of-mass can be repeated. It is easy to see that they have exactly the same transformation properties *in the phase space* that the quantities defined in Subsect. 4.3 had: They form a Lorentzian 4-vector $\mathtt{P}^{\underline{a}}$ and an anti-symmetric tensor $\mathtt{J}^{\underline{a}\,\underline{b}}$, and transform according to the Poincaré transformation. However, defining the Cartesian spacetime coordinates by $x^{\underline{a}} := (t, x^{\mathbf{i}})$, we can consider the transformation of $\mathtt{P}^{\underline{a}}$ and $\mathtt{J}^{\underline{a}\,\underline{b}}$ under the Poincaré transformation of the Cartesian coordinates, $x^{\underline{a}} \mapsto x^{\underline{b}} \Lambda_{\underline{b}}{}^{\underline{a}} + C^{\underline{a}}$, *in the spacetime* too. Using the explicit form of M and M^a in terms of the spacetime Cartesian coordinates and the defining equation (56), it is a straightforward calculation to show that $\mathtt{P}^{\underline{a}}$ and $\mathtt{J}^{\underline{a}\,\underline{b}}$ transform just in the correct way. It might be worth noting that the special *linear* time dependence of the centre-of-mass is needed to derive the correct transformation properties. In fact, the relativistic angular momentum tensor built from the Regge–Teitelboim angular momentum and the Beig–Ó Murchadha centre-of-mass does *not* transform in the expected way under Poincaré transformations of the Cartesian coordinates $x^{\underline{a}}$ *in the spacetime*.

Next let us consider again a general ξ^a, and calculate the *total* time derivative of $\mathtt{Q}[K^a]$ with respect to ξ^a. Now the coefficients in the asymptotic spacetime Killing vectors K^a have explicit time dependence. Using the evolution equations of Subsect. 3.2, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathtt{Q}\big[K^a\big] &= \mathtt{Q}\big[\dot{M} + M^e D_e N - N^e D_e M, \dot{M}^a + N D^a M - M D^a N - [N, M]^a\big] \\ &= 0 \end{aligned} \tag{57}$$

for any $K^a \in \mathcal{A}^K_\xi$. Therefore, *the energy-momentum and angular momentum, defined by* $\mathtt{Q}[K^a]$ *with the vector fields* K^a *that are asymptotic Killing with respect to* ξ^a*, are conserved in time provided the time evolution is defined by the same* ξ^a. Thus, just as in Subsects. 2.5 and 3.2, the vector field ξ^a defining the time evolution is required only to be an allowed time axis, but the generators K^a for the physical quantities do depend on ξ^a. In particular, both the conservation (35)–(38) of the time independent quantities with respect to gauge evolutions in Subsect. 4.3 and the conservation of the energy-momentum and relativistic angular momentum defined in the present subsection with respect to pure asymptotic time translations are special cases of (57).

7.3 Translations for Slow Fall-Off Metrics

In Subsects. 5.2 and 5.3 we saw that the asymptotic translations emerge as genuine asymptotic symmetries only for $1/r$ or faster fall-off, while for slow fall-off they are lost in the sea of the "generators of gauge evolutions" and the genuine asymptotic symmetries are only the asymptotic rotations and boosts. On the other hand, by the results of Subsect. 7.1, for slow, $1/r^k$, $1/2 < k < 1$, fall-off we can define energy-momentum but not relativistic angular momentum. The aim of the present subsection is to resolve this apparent contradiction by showing what the translations in the slow fall-off case might be.

The key observation is that $\mathtt{Q}[M, M^a]$ can be finite for the slow fall-off metrics provided the structure of M and M^a is

$$M\big(t, x^{\mathbf{k}}\big) = T\big(t\big) + r^K \mu^{(K)}\Big(t, \frac{x^{\mathbf{k}}}{r}\Big) + o^{\infty}\big(r^K\big)\,, \tag{58}$$

$$M_{\mathbf{i}}\big(t, x^{\mathbf{k}}\big) = T_{\mathbf{i}}\big(t\big) + r^L \mu^{(L)}_{\mathbf{i}}\Big(t, \frac{x^{\mathbf{k}}}{r}\Big) + o^{\infty}\big(r^L\big)\,, \tag{59}$$

where $K, L \leq -k$, i.e. the $x^{\mathbf{k}}$-dependent parts of M and M^a tend to zero as r^{-k} rather than diverging as $r^{(1-k)}$ as in (43) and (44). This motivates us to consider for some $q \leq (1-k)$ the spacetime vector fields $K^a = Mt^a + M^a$ whose asymptotic structure is given by (58)–(59) and $K, L \leq q$. We say that they have q-fast fall-off. In general, these vector fields do not form a Lie algebra.

Next consider the space ${}_q\mathcal{T}^K_\xi$ of such vector fields which are asymptotic Killing vectors too: Let the $t^a t^b \nabla_{(a}K_{b)}$ and $P^a_c t^b \nabla_{(a}K_{b)}$ parts of the Killing operator acting on them tend to zero at least as $O(r^{q-1})$. Then the Lie bracket $[K, \bar{K}]^a$ of $K^a \in {}_q\mathcal{T}^K_\xi$ and $\bar{K}^a \in \mathcal{A}^K_\xi$ contains terms of order r^{-k}. Thus the Lie bracket operation preserves the index q of the space ${}_q\mathcal{T}^K_\xi$ and the components of $[K, \bar{K}]^a$ have the structure (58)-(59) only if $q \geq -k$. The quotient ${}_q\mathcal{T}^K_\xi / {}_q\mathcal{T}^K_\xi \cap \mathcal{G}^K_\xi$ is isomorphic to $\mathbb{R}^4$ and inherits a commutative Lie algebra structure from $\mathcal{A}^K_\xi / \mathcal{G}^K_\xi$. Equations (47) and (48) show that $T(t)$ and $T_{\mathbf{i}}(t)$ are in fact constant for ξ^a generating e.g. an asymptotically collapsing or asymptotically Cartesian coordinate system. (If ξ^a generates an asymptotically Rindler coordinate system, then they still depend on time as $T(t) = T\cosh(2t) + T^*\sinh(2t)$ and $T_{\mathbf{i}}(t) = T_{\mathbf{i}} + \beta_{\mathbf{i}}(T^*\cosh(2t) + T\sinh(2t))$ for constants T, T^* and $T_{\mathbf{i}}$ satisfying $T_{\mathbf{i}}\beta^{\mathbf{i}} = 0$.) Thus ${}_q\mathcal{T}^K_\xi$ may be interpreted as the space of the "q-fast fall-off asymptotic translations" in $\mathcal{A}^K_\xi$ even if $k \in (0,1)$, provided $-k \leq q \leq (1-k)$. On the other hand, by the results of Subsect. 7.1 the translations yielding finite energy-momentum can be the elements of ${}_q\mathcal{T}^K_\xi$ for any $q \leq (1-k)$ if $k \geq 1$, but for $0 < k < 1$ only those "q-fast fall-off" translations yield finite energy-momentum for which $q \leq -k$. Therefore, the space of the fast fall-off translations yielding finite energy-momentum is precisely ${}_{-k}\mathcal{T}^K_\xi$.

8 Summary

The present investigation grew from the need to give a systematic *space-time* interpretation of the results and the main points of the analysis of canonical vacuum general relativity by Beig and Ó Murchadha. However, while the centre-of-mass components of the relativistic angular momentum of matter fields in Minkowski spacetime depend linearly on time, the Beig–Ó Murchadha centre-of-mass expression for asymptotically flat spacetimes is completely time independent. As a consequence of this the Beig–Ó Murchadha centre-of-mass is conserved only with respect to "gauge evolutions", and although it transforms in the correct way in the phase space, it does not in the spacetime.

To find the correct time dependence we suggest to parameterize the Beig–Ó Murchadha Hamiltonian by the lapse and shift parts of appropriately defined asymptotic Killing vector fields. A natural Lie algebra structure can be introduced on the quotient of the space of the asymptotic Killing fields and the subspace of "gauge generators", and we showed that this Lie algebra is only the Lorentz Lie algebra for slow fall-off, but it is the Poincaré algebra for $1/r$ or faster fall-off metrics.

We define the total energy-momentum and relativistic angular momentum by the value on the constraint surface of the Beig–Ó Murchadha Hamiltonian parameterized by the *asymptotic translation or rotation-boost Killing vectors*. This definition is completely analogous to that of the (quasi-local or total) energy-momentum and angular momentum of matter fields using the Killing vectors of the Minkowski spacetime. The energy-momentum obtained in this way is just the standard ADM energy-momentum and the spatial angular momentum is that of Regge and Teitelboim. However, the centre-of-mass deviates from that of Beig and Ó Murchadha by a term, which is the linear momentum times the coordinate time. This centre-of-mass has the correct transformation properties, known for the matter fields in flat spacetime, both in the phase space and in the spacetime with respect to asymptotic Poincaré transformations, and it is conserved if the time evolution is generated by asymptotic time translations.

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