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Causal Measurability in Chronological Spaces

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We show that the causal structure determines a volume measurability up to sets of zero measure. In space-time manifolds this causal measurability, apart from sets of zero measure, agrees with the a priori four-dimensional Lebesgue measurability, provided the strong causality condition holds.

1. INTRODUCTION

The standard model of space-time is a smooth manifold equipped with a Lorentz metric [1, 2]. The points of this manifold represent events, the Lorentz metric up to conformal factor describes the causal relations between them, and, finally, the conformal factor fixes the affine and metric properties. This model contains numerous mathematical (e.g., topological, analytical, measure theoretical, causal, geometrical) structures. We think, however, that from a physical point of view these structures have different significance: the deepest one is causality and some others are introduced for their mathematical usefulness only.

As is well-known, by mans of causal structure it is possible to define topologies. Such is the Alexandrov topology, which agrees with the manifold topology provided the strong causality condition holds. One can think this causal topology is the "true," the physical topology of the collection of events. Similarly, if the causal structure were able to determine some other mathematical structures, then these structures would be the deepest ones. It is expected that these new structures reflect the properties of causality and agree with the usual ones at least in the absence of certain causal pathologies.

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In this paper we examine the problem of volume measurability. Measurability, in the form of volume integrals on subsets of the space-time, appears in different areas of physics. Although the value of the integral or, more generally, the volume of a measurable set, is determined by physical quantities, e.g., the metric, the fact that a given set is measurable or not is independent of the physical properties of the events that make up the set. This measurability comes from Euclidean spaces, similar to the manifold topology. Although, based on the local $\mathbb{R}^4$ character of space-time, it seems reasonable to define the measurable subsets of the space-time as the ones inherited from $\mathbb{R}^4$, no direct physical experience exists implying that we have to do this: measurability is given a priori.

In the present paper we show that the chronology relation determines a measurability of the Lebesgue type. This causal measurability is unique up to sets of zero causal measure; more precisely, each measurable set with respect to the outer measure induced by a physically reasonable premeasure is a union of a set of zero outer measure and a set which is measurable with respect to the outer measure induced by any other physically reasonable premeasure. It will be shown that in a space-time manifold each causally measurable set, apart from sets of zero causal measure, is measurable with respect to the ordinary four-dimensional Lebesgue measure. Furthermore, if the strong causality condition holds, then each Lebesgue measurable set, apart from sets of zero Lebesgue measure, is causally measurable. Therefore, at least in strongly causal space-times, we have shown that, apart from sets of zero measure, the a priori Lebesgue measurability is justified and derived from causality.

In Section 2 we review the Lebesgue measurability in (space-time) manifolds, where we stress again that the ordinary Lebesgue measurability is a priori given. Section 3 is then main part of this article: it contains the construction and some of the properties of the causal measurability. In Section 4 we examine the relations between the causal and the Lebesgue measurabilities in space-time manifolds.

The notions of causality and measure theory we use are those of Refs. 1 and 3, respectively. In general, capital script letters denote collections of subsets of the base space M, while italic capital and small letters denote subsets and points of M, respectively.

2. MEASURABILITY IN (SPACE-TIME) MANIFOLD

Let M denote the set of events. We have to impose several assumptions on M, some are motivated by physical observations, others are purely mathematical to handle M. M is a connected, Hausdorff, paracompact topological space, homeomorphic locally to $\mathbb{R}^4$, on which a smooth atlas is given. (For the discussion of these assumptions see Refs. 1 and 2.)

The local $\mathbb{R}^4$ character of M defines a class of four-dimensional Lebesgue measures on M, being absolutely continuous [3] with respect to each other, and if the measure λ is fixed then an integration with respect to λ is also given. λ is naturally connected to the manifold topology: it is a so-called regular Borel measure [3].

Although the metric g, determining all the geometrical properties of space-time, has not been specified, the whole collection $\mathscr{L}$ of measurable sets of M is given. Thus $\mathscr{L}$ is independent of both the causal structure and the conformal factor. Thus one may raise the question of whether or not it is possible to determine measurability in a natural way by making use of the causal structure alone.

3. CAUSAL MEASURABILITY

In what follows we try to outline the beginnings of a causal measure theory. The idea we use is the construction of the n-dimensional Lebesgue measure: an application of the extension problem of measure theory [3]. The extension is made in two steps: (1) a nonnegative set function μ, defined on a class $\mathscr{H}$ of subsets of the base space M, induces an outer measure μ^* on the power set $\mathscr{P}(M)$ of M; (2) if $\mathscr{H}$ is at least a semiring and μ is additive and subadditive on $\mathscr{H}$, then the collection $\mathscr{M}$ of μ^*-measurable sets and the outer measure μ^* itself are extensions of $\mathscr{H}$ and μ, respectively.

3.1. Chronological Spaces

The construction of $\mathscr{H}$ is based on chronology; thus, we define the notion of chronological spaces [4–6] as follows: The pair $(M, \ll)$ is a chronological space if M is a nonempty set and $\ll$ is a transitive relation on M satisfying the following condition: if $x \ll x$, then there is a point $y \in M$ such that $y \neq x$, $x \ll y \ll x$. The points of M are the events and $\ll$ is called the chronology relation. A special but physically very important class of chronological spaces is called full [5]: $(M, \ll)$ is full if for $\forall p_1, p_2, x, q_1, q_2 \in M$ satisfying $p_1, p_2 \ll x \ll q_1, q_2$ there are points $p, q \in M$ such that $p_1, p_2 \ll p \ll x \ll q \ll q_1, q_2$; furthermore, for $\forall x \in M$ there are points $p, q \in M$ such that $p \ll x \ll q$.

The chronological future of the set $Z \subseteq M$ is $I^+Z := \{x \in M \mid \exists z \in Z: z \ll x\}$. A subset $S \subseteq M$ is called achronal if $S \cap I^+S = \emptyset$. The set $F \subseteq M$ is called future set [1] if $I^+F \subseteq F$ and the collection of future sets will be denoted by $\mathscr{I}^+$. Of course, every definition and statement has a dual form [5] interchanging the future and the past. Based on the definitions one can show easily the next proposition.

Proposition 3.1.1. (1) $F \in \mathscr{I}^+$ iff $(M-F) \in \mathscr{I}^-$; (2) $M, \emptyset \in \mathscr{I}^+$; (3) for arbitrary collection $\{F_\gamma | \gamma \in \Gamma\}$ of future sets we have

$$I^+ \bigcap \{F_\gamma | \gamma \in \Gamma\} \subseteq \bigcap \{F_\gamma | \gamma \in \Gamma\}$$

i.e., the intersection of arbitrary collection of future sets is a future set (Γ is an arbitrary indexing set); (4) the union of arbitrary collection of future sets is a future set.

By means of $\ll$ a reasonable topology can be defined on M: The Alexandrov topology [5] is the coarsest topology $\mathscr{T}$ on M in which each $I^\pm Z$ is open for $\forall Z \subseteq M$. One can show that the interior, $\operatorname{int} F$, and the closure $\bar{F}$ of a future set F, with respect to $\mathscr{T}$, are also future sets.

Full chronological spaces have simple topological properties [5]. The collection $\mathscr{C}$ of intervals $\langle u, v\rangle := I^+\{u\} \cap I^-\{v\}$, $u \ll v$, is a base of $\mathscr{T}$, and for $\forall F \in \mathscr{I}^+$ $\operatorname{int} F = I^+F$, $\bar{F} = \{x \in M | I^+\{x\} \subseteq F\}$. While in an arbitrary topological space the existence of a countable dense subset of the base space (i.e., the separability) does not imply the existence of a countable base; in full chronological spaces these concepts coincide.

3.2. The Chronological Semiring

Since we want to build up a construction similar to the Lebesgue measure on $\mathbb{R}^n$, we need a class of subsets of M, defined only by means of $\ll$, having a ring or semiring structure [3]. The next proposition shows that the collection $\mathscr{H} := \{F \cap P | F \in \mathscr{I}^+, P \in \mathscr{I}^-\}$ is a δ-semiring with identity element M.

Proposition 3.2.1. (1) $M, \emptyset \in \mathscr{H}$; (2) for arbitrary collection $\{H_\gamma | \gamma \in \Gamma\}$ of sets from $\mathscr{H}$ we have

$$\bigcap \{H_\gamma | \gamma \in \Gamma\} \in \mathscr{H}$$

(3) If $H, H_1 \in \mathscr{H}$ such that $H_1 \subseteq H$ then there exist sets $H_2, H_3 \in \mathscr{H}$ satisfying $H_1 \cap H_2 = H_1 \cap H_3 = H_2 \cap H_3 = \emptyset$ and $H_1 \cup H_2 \cup H_3 = H$.

Proof. (1) follows trivially from Proposition 3.1.1. For $\forall H_\gamma$, $\gamma \in \Gamma$, there exist sets $F_\gamma \in \mathscr{I}^+$, $P_\gamma \in \mathscr{I}^-$ such that $H_\gamma = F_\gamma \cap P_\gamma$. Hence, because of Proposition 3.1.1, we have $\bigcap \{H_\gamma | \gamma \in \Gamma\} = \bigcap \{F_\gamma | \gamma \in \Gamma\} \bigcap \{P_{\gamma'} | \gamma' \in \Gamma\} \in \mathscr{H}$. For the sets $H, H_1 \in \mathscr{H}$ satisfying $H_1 \subseteq H$ there exist sets $F_1 \in \mathscr{I}^+$, $P_1 \in \mathscr{I}^-$ such that $H_1 = F_1 \cap P_1 \subseteq H$. Then the sets $H_2 := H \cap (M - F_1)$, $H_3 := H \cap F_1 \cap (M - P_1)$ belong to $\mathscr{H}$; H_1, H_2, H_3 are disjoint sets and their union is H. ∎

$\mathscr{H}$ is called the chronological semiring. To characterize minimal elements of $\mathscr{H}$, it is useful to introduce the concept of atom, borrowed from lattice theory: The nonempty set $A \in \mathscr{H}$ is called an atom if for every set $H \subseteq A$, $H \in \mathscr{H}$ implies either $H = \emptyset$ or $H = A$. If A is an atom and $H \in \mathscr{H}$, then either $A \subseteq H$ or $A \cap H = \emptyset$. Of course, a set containing a single point and belonging to $\mathscr{H}$ is an atom and is called a trivial atom. Conversely, every achronal set containing at least two points cannot be an atom.

The following statement gives us the physical meaning of nontrivial atoms: they are irreducible parts of the chronology violating sets [6].

Proposition 3.2.2. A set A is an atom containing at least two points iff $A = \langle x, x\rangle$ for some $x \in A$.

Proof. If for some $x \in A$, $A = \langle x, x\rangle$, then A, according to the properties of $\ll$, contains at least two points. If H is a nonvoid subset of $\langle x, x\rangle$ belonging to $\mathscr{H}$, then $\exists F \in \mathscr{I}^+$, $P \in \mathscr{I}^-$ such that $H = F \cap P$. Then, for $\forall z \in P \cap F \subseteq \langle x, x\rangle$, $\langle x, x\rangle = \langle z, z\rangle = I^+\{z\} \cap I^-\{z\} \subseteq I^+F \cap I^-P \subseteq F \cap P \subseteq \langle x, x\rangle$; i.e., $H = \langle x, x\rangle = A$ is an atom.

Let A be an atom containing at least two points and let $F \in \mathscr{I}^+$, $P \in \mathscr{I}^-$ such that $A = F \cap P$. Then $\exists x \in A$ for which $P \cap I^+\{x\} \neq \emptyset$, because otherwise A would be achronal: $A \cap I^+A \subseteq P \cap I^+A = P \cap \bigcup \{I^+(x) | x \in A\} = \bigcup \{P \cap I^+(x) | x \in A\} = \emptyset$. Hence, $P \cap I^+\{x\} \neq \emptyset$, which belongs to $\mathscr{H}$, and $P \cap I^+\{x\} \subseteq P \cap I^+A \subseteq P \cap I^+F \subseteq P \cap F = A$. But A is an atom; thus, $x \in A = A \cap I^+\{x\}$, implying $\langle x, x\rangle \neq \emptyset$ and $\langle x, x\rangle \cap A \neq \emptyset$. Because of the first part of this proposition $\langle x, x\rangle$ is an atom too, $\langle x, x\rangle = A$. ∎

In the rest of this section some topological properties of $\mathscr{H}$ are given.

Proposition 3.2.3. If $H \in \mathscr{H}$, then $\operatorname{int} H$, $\bar{H} \in \mathscr{H}$.

Proof. If $H = F \cap P$ for some $F \in \mathscr{I}^+$ and $P \in \mathscr{I}^-$ then, because of $\operatorname{int} F \in \mathscr{I}^+$, $\operatorname{int} P \in \mathscr{I}^-$, $\operatorname{int} H = \operatorname{int} F \cap \operatorname{int} P \in \mathscr{H}$. From $F \cap P \subseteq \bar{F} \cap \bar{P}$ it follows that $\bar{H} \subseteq \bar{F} \cap \bar{P}$. Then $I^+\bar{H} \subseteq I^+(\bar{F} \cap \bar{P}) \subseteq I^+\bar{F} = I^+F \subseteq F$ and similarly $I^-\bar{H} \subseteq P$. If $F' := F \cup \bar{H}$ and $P' := P \cup \bar{H}$, then $I^+F' = I^+F \cup I^+\bar{H} \subseteq F \subseteq F'$ and $I^-P' \subseteq P'$; i.e., $F' \in \mathscr{I}^+$ and $P' \in \mathscr{I}^-$. Since $H \subseteq \bar{H}$, it follows that $\bar{H} = (F \cap P) \cup \bar{H} = F' \cap P' \in \mathscr{H}$. ∎

Corollary. For $\forall H \in \mathscr{H}$ there are disjoint empty-interior sets $\Delta_1 H$, $\Delta_2 H \in \mathscr{H}$ such that $\operatorname{int} H \cap \Delta_1 H = \operatorname{int} H \cap \Delta_2 H = \varnothing$, $\operatorname{int} H \cup \Delta_1 H \cup \Delta_2 H = H$ and $\Delta_1 H \cup \Delta_2 H \subseteq \partial H$. If $H = \bar{H}$, then $\Delta_1 H \cup \Delta_2 H = \partial H$.

Proof. If $H \in \mathscr{H}$ then, as in the proof of Proposition 3.2.1, for $\operatorname{int} H$ define $\Delta_1 H := H \cap (M - \operatorname{int} F)$, $\Delta_2 H := H \cap \operatorname{int} F \cap (M - \operatorname{int} P)$. Then $\operatorname{int} \Delta_1 H = \operatorname{int} F \cap \operatorname{int} P \cap \operatorname{int}(M - \operatorname{int} F) \subseteq \operatorname{int} F \cap \operatorname{int} P \cap (M - \operatorname{int} F) = \varnothing$ and similarly $\operatorname{int} \Delta_2 H \subseteq \varnothing$. A simple calculation shows that $\Delta_1 H \cup \Delta_2 H = H \cap \overline{(M-H)}$; thus if $H = \bar{H}$, then $\Delta_1 H \cup \Delta_2 H = \partial H$. ▮

Proposition 3.2.4. Let $(M, \ll)$ be a full chronological space. For the subset $H \subseteq M$, $H \cap I^+ H = \varnothing$ iff $H \in \mathscr{H}$ and $\operatorname{int} H = \varnothing$.

Proof. Let $H \in \mathscr{H}$, $\operatorname{int} H = \varnothing$ and, on the contrary, suppose $\exists z, w \in H$ such that $z \ll w$. But $H = F \cap P$ for some $F \in \mathscr{I}^+$ and $P \in \mathscr{I}^-$; thus $\varnothing \neq \langle z, w \rangle \subseteq I^+ F \cap I^- P \subseteq F \cap P = H$, contradicting the hypothesis $\operatorname{int} H = \varnothing$.

If $H \cap I^+ H = \varnothing$, then $F := H \cup I^+ H \in \mathscr{I}^+$, $P := H \cup I^- H \in \mathscr{I}^-$, and $H = F \cap P$; i.e., $H \in \mathscr{H}$. If there were points $u, v \in M$ such that $\varnothing \neq \langle u, v \rangle \subseteq H$ then, because of the fullness of $(M, \ll)$, there would be points $x, y \in M$ satisfying $u \ll x \ll y \ll v$; which would imply $x, y \in H$. ▮

3.3. The Causal Measurability

Every set function μ: $\mathscr{H} \to [0, \infty]$ defines a (subadditive) outer measure μ^*: $\mathscr{P}(M) \to [0, \infty]$, but, in general, the elements of $\mathscr{H}$ are not μ^* measurable, and for $H \in \mathscr{H}$, $\mu^*(H)$ is not necessarily equal to $\mu(H)$. However, if μ is additive and subadditive on $\mathscr{H}$, then these difficulties cannot occur [3].

Each of our observations and measurements is a localized event or process. This means, within the framework of measure theory, that the event or process is contained in a region with finite measure. Thus, if some region had infinite measure we would expect to have a countable covering of it consisting of sets of finite measure; i.e., the σ finiteness [3] of the measure is expected.

A set function μ: $\mathscr{H} \to [0, \infty]$ which is additive, subadditive, and σ finite on $\mathscr{H}$ is called a causal premeasure. One can show that if $(M, \ll)$ is full then a causal premeasure always exists. (For a constructive proof see Appendix 3 of an earlier, preprint version of this paper [7].)

If $\mathscr{B}(\mathscr{D})$ denotes the σ algebra generated by $\mathscr{D}$, $\mathscr{D} \subseteq \mathscr{P}(M)$, then the next statement, which is a direct consequence of the theorems of standard measure theory [3] and the properties of $\mathscr{H}$ and μ, lists the main properties of the causal measure and measurability.

Theorem 3.3.1. Let $(M, \ll)$ be full, let μ^* denote the outer measure on M induced by the premeasure μ, and let $\mathscr{M}$ be the collection of μ^*-measurable subsets of M. Then (1) $\mu^*(N) = 0$ implies $N \in \mathscr{M}$, (2) $\mathscr{M}$ is a σ algebra with identity element M and

$$\mathscr{M} = \overline{\overline{\mathscr{B}(\mathscr{H})}} := \{B \cup N \mid B \in \mathscr{B}(\mathscr{H}), \mu^*(N) = 0\}$$

(3) μ^* is subadditive, additive, and σ finite on $\mathscr{M}$; (4) for $\forall H \in \mathscr{H}$, $\mu(H) = \mu^*(H)$; (5) for $\forall X \subseteq M$, $\exists E \in \mathscr{M}$: $X \subseteq E$ and $\mu^*(X) = \mu^*(E)$. μ^* is called a causal outer measure, and the restriction $\bar{\mu}$ of μ^* to $\mathscr{M}$ is a causal measure; the elements of $\mathscr{M}$ are called causally measurable subsets of M.

As Theorem 3.3.1 shows, each element of $\mathscr{H}$ is causally measurable and $\bar{\mu}$ is an extension of μ from $\mathscr{H}$ to $\mathscr{M}$. Let μ_1, μ_2 be causal premeasures with the corresponding causal measures $\bar{\mu}_1$, $\bar{\mu}_2$, and σ algebras $\mathscr{M}_1$, $\mathscr{M}_2$, respectively. If $E \in \mathscr{M}_1$ then, according to the theorem above, $\exists B \in \mathscr{B}(\mathscr{H})$ and $N \subseteq M$ such that $E = B \cup N$ and $\bar{\mu}_1(N) = 0$. Then $B \in \mathscr{M}_2$ too, and $\bar{\mu}_1(B) \leqslant \bar{\mu}_1(B \cup N) \leqslant \bar{\mu}_1(B) + \bar{\mu}_1(N) = \bar{\mu}_1(B)$ which implies $\bar{\mu}_1(E) = \bar{\mu}_1(B)$. Thus for each $E_1 \in \mathscr{M}_1$ there is a set $E_2 \in \mathscr{M}_2$ such that $E_2 \subseteq E_1$ and $\bar{\mu}_1(E_1) = \bar{\mu}_1(E_2)$. Causal measurability is, therefore, unique up to sets of zero causal measure.

A nonmeasurable subset of M is, for example, a proper subset X of a nontrivial atom $\langle x, x \rangle$ having positive masure: for the test set $T := \langle x, x \rangle$ $\mu^*(T) = \mu^*(\langle x, x \rangle) = \mu^*(T \cap X) = \mu^*(T - X)$, implying $\mu^*(T) \neq \mu^*(T \cap X) + \mu^*(T - X)$.

3.4. Topological Conditions

Without further restrictions, there might be situations in which a chronologically extended region has zero measure, while the measure of the boundary of a closed or open subset of M is positive. We expect, however, that such situations cannot occur.

To rule out these topological pathologies of the measure, we have to impose certain topological conditions. We will say that the set function μ: $\mathscr{D} \to [0, \infty]$, defined on a collection $\mathscr{D} \subseteq \mathscr{P}(M)$ of subsets of the topological space $(M, \mathscr{T})$, satisfies the first topological condition if $\mu(D) = 0$ implies $\operatorname{int} D = \varnothing$, for $D \in \mathscr{D}$, and satisfies the second topological condition if $\operatorname{int} D = \varnothing$ implies $\mu(D) = 0$, for $D \in \mathscr{D}$. The next statement shows if the causal premeasure satisfies the first topological condition, then the first of the pathologies cannot occur.

Proposition 3.4.1. The causal outer measure μ^* satisfies the first topological condition iff the causal premeasure μ does.

Proof. μ^* is an extension of μ; thus, if μ^* satisfies the first topological condition, μ does.

Conversely, suppose, on the contrary, that for some $X \subseteq M$, int $X \neq \varnothing$ and $\mu^*(X)=0$. Then $\exists \langle u, v \rangle \in \mathscr{C}$ such that $\langle u, v \rangle \subseteq X$. But μ^* is an extension of μ [i.e., for $\forall H \in \mathscr{H}$, $\mu(H)=\mu^*(H)$]; thus, we have $\mu[\langle u, v \rangle] \leqslant \mu^*(X)=0$, contradicting the first topological condition. ▮

μ^* may not satisfy the second topological condition even if μ satisfies both of the topological conditions: for the subset X at the end of the preceding paragraph int $X=\varnothing$ but $\mu^*(X)=\mu(\langle x, x \rangle)>0$. In fact, we would like to require the topological conditions only for $\bar{\mu}$ and not for μ^*. Trivially, to ensure the existence of such a causal measure $\bar{\mu}$, the existence of a causal premeasure μ satisfying both of the topological conditions is necessary, but we do not know if it is sufficient or not.

4. CAUSAL MEASURABILITY IN SPACE-TIME MANIFOLD

If (M, g) is a space-time manifold, then $(M, \ll)$ is a full chronological space with the natural chronological relation $\ll$ defined by g[1, 2, 4–6, 8]. Thus, on (M, g) there are two measurabilities: the causal and the four-dimensional Lebesgue measurabilities. In this section, we examine the relations between these two.

Theorem 4.1. In a space-time manifold $\mathscr{B}(\mathscr{H}) \subseteq \mathscr{L}$.

Proof. For $\forall F \in \mathscr{I}^+$, the boundary $\partial^m F$ of F (with respect to the manifold topology $\mathscr{T}^m$) is a C^{1-} hypersurface [1]; thus, each subset of $\partial^m F$ has zero Lebesgue outer measure and, as a consequence, is Lebesgue-measurable. But each $\mathscr{T}^m$-open set is also Lebesgue measurable; therefore, $F=\text{int}^m F \cup (F \cap \partial^m F) \in \mathscr{L}$, which implies $\mathscr{I}^+ \subseteq \mathscr{L}$ ($\text{int}^m F$ denotes the interior of F with respect to $\mathscr{T}^m$). Similarly, $\mathscr{I}^- \subseteq \mathscr{L}$; therefore, $\mathscr{B}(\mathscr{H}) \subseteq \mathscr{L}$. ▮

Theorem 4.1 implies that for each causally masurable set $E \in \mathscr{M}$ there is a set $E' \in \mathscr{M}$ being Lebesgue measurable, too, and $E' \subseteq E$, such that $\bar{\mu}(E-E')=0$. Thus every causally measurable subset of M for arbitrary causal premeasure, apart from sets of zero causal masure, is Lebesgue measurable.

The next statement shows that the converse is also true provided the strong causality condition [1, 8] holds.

Theorem 4.2. If (M, g) satisfies the strong causality condition, then $\mathscr{B}(\mathscr{T}^m) \subseteq \mathscr{B}(\mathscr{H}) \subseteq \mathscr{M}$.

Proof. If (M, g) is strongly causal, then [1, 5, 8] $\mathscr{T}^m=\mathscr{T}$; therefore, $(M, \mathscr{T})$ is second-countable. Thus there is a countable subcollection $\mathscr{C}_0 \subseteq \mathscr{C}$ which is a base of $\mathscr{T}^m$. Then $\mathscr{T}^m \subseteq \mathscr{B}(\mathscr{C}_0) \subseteq \mathscr{B}(\mathscr{H})$, implying $\mathscr{B}(\mathscr{T}^m) \subseteq \mathscr{B}(\mathscr{H})$. ▮

One can construct space-times, with no closed timelike but with closed nonspacelike curves, in which there is a subset belonging to $\mathscr{B}(\mathscr{T}^m)$ but to $\mathscr{B}(\mathscr{H})$, and space-times as above in which $\mathscr{B}(\mathscr{T}^m) \subseteq \mathscr{B}(\mathscr{H})$. These examples suggest that the chronological condition cannot ensure $\mathscr{B}(\mathscr{T}^m) \subseteq \mathscr{B}(\mathscr{H})$, but the causality condition is not necessary.

5. CONCLUSION AND REMARKS

By means of the chronology relation we were able to create a δ-semiring $\mathscr{H}$ with identity element M. Every nonnegative set function μ on $\mathscr{H}$ defines an outer measure μ^* and a measurability with respect to μ^*. If μ is additive, subadditive, and σ finite (such a μ always exists in full chronological spaces and is called causal premeasure), then μ^* will be an extension of μ and every set $H \in \mathscr{H}$ will be μ^*-measurable. If μ_1, μ_2 are causal premeasures and E_1 is μ_1^*-measurable, then there is a set $E_2 \subseteq E_1$ being μ_1^*- and μ_2^*-measurable such that $\mu_1^*(E_1-E_2)=0$; i.e., causal measurability is unique up to sets of zero measure. In space-time manifolds every causally measurable subset, apart from sets of zero causal measure, is Lebesgue measurable and vice-versa, provided the strong causality condition holds. Thus, observing the chronological relation in a strongly causal space-time, Lebesgue measurability, apart from sets of zero Lebesgue measure, is recovered; i.e., it is derived from causality. Hence we were able to reduce the number of unjustified assumptions in the mathematical model of space-time.

One of the most interesting open questions is what kind of conditions are able to guarantee the uniqueness of the causal measurability. For example, can physically reasonable topological conditions on the premeasure be required such that every such premeasure gives the same measurability; i.e., $\mu_1^*(N)=0$ iff $\mu_2^*(N)=0$ for every two such premeasure μ_1, μ_2.

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