Commutation properties of cyclic and null Killing symmetries

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In a space-time admitting cyclic and nonspacelike Killing symmetries the commutation properties of the Killing vectors are examined. It is shown that cyclic and null Killing vectors can be noncommuting only if a Killing vector of stationarity is also admitted. Two consequences of this commutativity are also discussed.

I. INTRODUCTION

Stationary, axisymmetric space-times have distinguished importance in general relativity, e.g., the final state of black holes is thought of as stationary and axisymmetric.[1] It is usually assumed that the Killing vectors of stationarity and axisymmetry commute. In fact, Carter[2] has shown that, without loss of any generality, this can always be assumed.

Axisymmetric space-times with null Killing vectors may also have physical significance; e.g., certain *pp* waves[3] or the Lukács–Perjés–Sebestyén solution[4] (which describes the gravitational field of a zero-mass, spinning charged particle) have these symmetries. Recently Lessner[5] has proposed certain axisymmetric vacuum solutions, admitting null Killing symmetry, of the five-dimensional Einstein equations as models of extended massless particles. The commutation of these Killing vectors is also assumed. Unfortunately, this commutation property does not follow from Carter's theorem.

In the present paper we generalize Carter's theorem. No fixed point is needed, so the axial symmetry is weakened to cyclic symmetry; and the timelike Killing symmetry is replaced by a nonspacelike one. We show that in a cyclically symmetric space-time, admitting a null Killing vector field, the two Killing vectors must commute, unless otherwise, in addition, the space-time has to admit a timelike Killing symmetry, too. Finally, based on this commutation property, we give a sufficient condition on a cyclically and null Killing symmetric space-time to be in Kundt's class[3] and it is shown that in space-times describing axial symmetric *pp* waves the null Killing vector must be orthogonal to the orbits of axial symmetry.

By space-time we mean a smooth, paracompact four-dimensional manifold M endowed with a Lorentzian metric,[1] but we do not use any field equation.

II. CYCLICALLY SYMMETRIC SPACE-TIME WITH NONSPACELIKE KILLING SYMMETRY

Space-time (M,g) is said to be cyclically symmetric[2] if there is a smooth map σ: $SO(2)\times M\to M$: $(\varphi,p)\mapsto\sigma(\varphi,p)$ for which each of the following conditions holds: (1) $\forall\varphi\in SO(2)$ the map $\sigma(\varphi): M\to M: p\mapsto\sigma(\varphi,p)$ is an isometry of (M,g); (2) $\forall\varphi',\varphi\in SO(2)$, $\sigma(\varphi)\circ\sigma(\varphi')=\sigma(\varphi+\varphi')$; (3) if $\sigma(\varphi)=\mathrm{Id}_M$ then $\varphi=0$ [i.e., SO(2) acts on M effectively]; and (4) the vector $X_p:=(\partial/\partial\varphi)_{\sigma(\varphi,p)|\varphi=0}$ is spacelike $\forall p\in M$.

One can define the orbit through p as $O(p):=\{\sigma(\varphi,p)\,|\,\varphi\in SO(2)\}$, and p is said to be a fixed point if $O(p)=\{p\}$. It is easy to show that p is a fixed point iff $X_p=0$, and if p is not a fixed point then there is a diffeomorphism of SO(2) onto $O(p)$ and so X, defined pointwise by $p\mapsto X_p$, is a smooth vector field. Here X will be called a cyclic Killing vector field.

Proposition 1: Let (M,g) be cyclically symmetric with SO(2) action σ and cyclic Killing vector field X, and let K be a nowhere vanishing future directed smooth nonspacelike Killing vector field on M. Then the vector field $\widetilde{K}$, defined pointwise by

$$\widetilde{K}_p:=\frac{1}{2\pi}\int_0^{2\pi}(\sigma(\varphi)_*K)_p\,d\varphi,\quad p\in M,$$

is a future directed nowhere vanishing smooth nonspacelike Killing vector field, which is invariant under the action σ; i.e., $[X,\widetilde{K}]=0$.

Proof: Since $\forall\varphi\in SO(2)$, $\sigma(\varphi)$ is an isometry, thus $\sigma(\varphi)_*K$ is a nowhere zero smooth nonspacelike Killing vector field. Here M is time oriented, therefore $\sigma(\varphi)_*K$ is also future directed. Consequently, $\widetilde{K}$ is a nowhere vanishing future directed nonspacelike smooth Killing vector field. $\forall\varphi\in SO(2)$

$$\sigma(\varphi)_*\widetilde{K}=\frac{1}{2\pi}\int_0^{2\pi}(\sigma(\varphi+\varphi')_*K)d\varphi'=\frac{1}{2\pi}\int_0^{2\pi}(\sigma(\varphi')_*K)d\varphi'=\widetilde{K},$$

i.e., $\widetilde{K}$ is invariant under the action σ. However, because of Corollary 1.8 and 1.11 of Ref. 6, this is equivalent to $[X,\widetilde{K}]=0$. □

Recall that the space-time is said to be stationary if it admits a nowhere vanishing smooth timelike Killing vector field.

Corollary: Let (M,g) be stationary and cyclically symmetric with cyclic Killing vector field X. Then there is a future directed smooth timelike vector field V which commutes with X.

This statement is a generalization of Carter's theorem[2]: it guarantees the existence of a timelike Killing vector field commuting with that of the cyclic symmetry in every stationary cyclically symmetric space-time, even in the presence of wire singularity. The existence of fixed points of σ is not needed; moreover, no restriction is required for the dimension of the space-time: it can be used for higher-dimensional Lorentzian geometries (e.g., in Kaluza–Klein theories) too.

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If K is timelike then $\widetilde{K}$ must be timelike too. If, however, K is nonspacelike or null, then $\widetilde{K}$ may be timelike on an (open) set and null on its complement. In the rest of this section, where the space-time is assumed to be cyclically and nonspacelike Killing symmetric with SO(2) action σ and Killing fields X and K, respectively, the causal character of $\widetilde{K}$ will be considered in fixed points of σ and along orbits diffeomorphic with SO(2) as well.

Proposition 2: If p is a fixed point of σ, then $\widetilde{K}$ is null at p iff K is null at p and $[X,K]$ vanishes at p.

Proof: Here p is fixed, thus $\forall\varphi\in \mathrm{SO}(2)$, $\sigma(\varphi,p)=p$ and $\sigma(\varphi)_*:T_pM\to T_pM$. Here $\widetilde{K}$ can be null at p only if K is null at p and there is a positive smooth function $f(\varphi)$ such that $\sigma(\varphi)_*K_p=f(\varphi)K_p$. This implies $f(\varphi+\varphi')=f(\varphi)f(\varphi')$, $\forall\varphi,\varphi'\in\mathrm{SO}(2)$. Its solution is $f(\varphi)=\exp(f'(0)\varphi)$. But $f(\varphi)=f(2\pi+\varphi)$ must hold, thus $f'(0)=0$ and $\sigma(\varphi)_*K_p=K_p$; i.e., $[X,K]$ vanishes at p.

Conversely, if K_p is null at p and $[X,K]$ vanishes at p then $\sigma(\varphi)_*K_p=K_p$ for $\forall\varphi\in\mathrm{SO}(2)$ and $\widetilde{K}_p=K_p$ is null. □

Now suppose p is not a fixed point of σ. Then $\forall\ q\in O(p)$ there is a unique element $\psi\in\mathrm{SO}(2)$ for which $q=\sigma(\psi,p)$. The following statement gives necessary and sufficient conditions that guarantee $\widetilde{K}$ being null along the orbit $O(p)$.

Proposition 3: The vector field $\widetilde{K}$ is null along the orbit $O(p)$ iff K is null along $O(p)$ and there is a smooth positive function $\Phi(\psi)$ for which $[X,K]_q=-\Phi(\psi)K_q$, $q=\sigma(\psi,p)$. [For such a $\Phi(\psi)$ the integral $\int_0^{2\pi}\Phi(\psi)d\psi$ is necessarily zero.]

Proof: The vector field $\widetilde{K}$ can be null at $q\in O(p)$ only if K is null all along $O(p)$ and there is a smooth positive function $f(\psi,\varphi)$ for which

$$(\sigma(\varphi)_*K)_q=f(\psi,\varphi)K_q\ .$$

This implies

$$f(\psi,\varphi+\varphi')=f(\psi,\varphi')f(\psi-\varphi',\varphi)\ , \tag{1}$$

$\forall\varphi,\varphi'\in\mathrm{SO}(2)$ and $q\in O(p)$. Let $F(\psi,\varphi):=\ln f(\psi,\varphi)$ and, denoting the derivative of F with respect to its first and second argument by F_1 and F_2, respectively, one obtains

$$F_2(\psi,\varphi+\varphi')=F_2(\psi,\varphi')-F_1(\psi-\varphi',\varphi)\ , \tag{2}$$

$$F_2(\psi,\varphi+\varphi')=F_2(\psi-\varphi',\varphi)\ . \tag{3}$$

The solution of Eq. (3) must have the form

$$F_2(\psi,\varphi)=\Phi(\psi-\varphi)\ , \tag{4}$$

where Φ is a smooth function. Using this expression, Eq. (2) yields

$$F_1(\psi,\varphi)=\Phi(\psi)-\Phi(\psi-\varphi)\ . \tag{5}$$

The integrability conditions for the system of partial differential equations (4) and (5) hold identically, and its solution is

$$F(\psi,\varphi)=\int_{\psi-\varphi}^{\psi}\Phi(u)du+F_0\,. \tag{6}$$

Substituting (6) into Eq. (1) one obtains $F_0=0$. Since f is periodic, i.e., $f(\psi,\varphi+2\pi)=f(\psi,\varphi)$, it follows that

$$\int_{\psi}^{\psi+2\pi}\Phi(u)du=0\ ,$$

$\forall\psi\in[0,2\pi]$, from which $\Phi(\psi+2\pi)=\Phi(\psi)$. This condition guarantees $f(\psi,\varphi)=f(\psi+2\pi,\varphi)$, too. According to Proposition 1, the vector field $\widetilde{K}$ is invariant under the action σ, thus

$$\begin{aligned}0&=2\pi[X,\widetilde{K}\,]\\&=\int_0^{2\pi}f(\psi,\varphi)d\varphi\,[X,K\,]+\frac{d}{d\psi}\int_0^{2\pi}f(\psi,\varphi)d\varphi\,K\\&=\int_0^{2\pi}f(\psi,\varphi)d\varphi\,[X,K\,]\\&\quad+\int_0^{2\pi}f(\psi,\varphi)(\Phi(\psi)-\Phi(\psi-\varphi))d\varphi\,K\,.\end{aligned}$$

But

$$\begin{aligned}&\int_0^{2\pi}f(\psi,\varphi)\Phi(\psi-\varphi)d\varphi\\&\quad=-\int_0^{2\pi}\frac{\partial f}{\partial\varphi}d\varphi=-f(\psi,2\pi)+f(\psi,0)=0\,,\end{aligned}$$

thus

$$[X,K\,]=-\Phi(\psi)K\,.$$

Conversely, if there is a smooth function $\Phi(\psi)$ for which $[X,K]=-\Phi(\psi)K$, then in the coordinate system (x^0,x^1,x^2,ψ) adapted to X, where the orbits are given by $x^0,x^1,x^2=\text{const}$,

$$K^a_q=K^a_p\exp\Big(-\int_0^{\psi}\Phi(u)du\Big)\,.$$

But $p=\sigma(2\pi,p)$, thus $K^a_{\sigma(2\pi,p)}$ must be equal to K^a_p. This implies $\int_0^{2\pi}\Phi(u)du=0$. The action of $\sigma(\varphi)_*$ on K can be calculated in the coordinate system (x^0,x^1,x^2,ψ):

$$\begin{aligned}(\sigma(\varphi)_*K)^a_q&=K^a_{\sigma(-\varphi,q)}\\&=K^a_p\exp\Big(-\int_0^{\psi-\varphi}\Phi(u)du\Big)\\&=K^a_q\exp\int_{\psi-\varphi}^{\psi}\Phi(u)du\,.\end{aligned}$$

Thus if K is null along $O(p)$ then

$$\widetilde{K}_q=\frac{1}{2\pi}\int_0^{2\pi}\Big(\exp\int_{\psi-\varphi}^{\psi}\Phi(u)du\Big)d\varphi\,K_q$$

is also null for $\forall q\in O(p)$. □

Corollary: Let $q=\sigma(\psi,p)$ be a point of $O(p)$, where $g(X,K)\neq0$. Then $\widetilde{K}$ is null along $O(p)$ iff K is null and $[X,K]=0$ along $O(p)$.

Proof: Since K is a Killing vector field and $\forall\varphi\in\mathrm{SO}(2)$, $\sigma(\varphi)$ is an isometry, $\sigma(\varphi)_*K$ is a null Killing vector field. Thus along $O(p)$ one has

$$\begin{aligned}0&=X^a(\sigma(\varphi)_*K)_{a;b}X^b=f(\psi,\varphi)X^aK_{a;b}X^b+X^aK_a\frac{\partial f}{\partial\psi}\\&=X^aK_af(\psi,\varphi)(\Phi(\psi)-\Phi(\psi-\varphi))\,.\end{aligned}$$

Since X^aK_a is not zero at $q=\sigma(\psi,p)$, $\Phi(\psi)=\Phi(\psi-\varphi)$, $\forall\varphi\in\mathrm{SO}(2)$; i.e., $\Phi=\text{const}$. But the only constant function having zero integral on $[0,2\pi]$ is the zero, thus $f(\psi,\varphi)=1$, $\forall\varphi,\psi\in\mathrm{SO}(2)$; i.e., $[X,K]=0$ along $O(p)$. □

Thus K is null along $O(p)$ iff $[X,K]=0$ and K is null, except the very special case in which X and K are orthogonal all along $O(p)$.

Although, if $X^a K_a = 0$ along $O(p)$, $\Phi(\psi)$ may be nonzero even if both K and $\widetilde{K}$ are null, but then, as the next proposition shows, the whole orbit $O(p)$ lies in the closure of an open set on which $\widetilde{K}$ is timelike.

Proposition 4: If the commutator $[X,K]$ does not vanish at some point $q \in O(p)$, then every neighborhood of each point of $O(p)$ contains a point where $\widetilde{K}$ is timelike.

Proof: If K is timelike at some point of $O(p)$, or if K is null along $O(p)$ but there is no function Φ required in Proposition 3, then $\widetilde{K}$ is timelike on $O(p)$. Thus one can assume that K is null and $[X,K] = -\Phi K$ along $O(p)$ for some smooth function $\Phi(\psi)$.

Let $r \in O(p)$ and W be a neighborhood of r. If K is not null at some point $s \in W$, then $\widetilde{K}$ is timelike on $O(s)$. Then one can assume that K is null on W. If there is a point $s \in W$ where the vector $[X,K]_s$ is not proportional to K_s, then a function Φ_s, required in Proposition 3, could not exist along the orbit $O(s)$, thus $\widetilde{K}$ is timelike on $O(s)$. One can assume therefore that $[X,K]$ is proportional to K on W. It will be shown, however, that K being null on W and $[X,K]$ being proportional to K on W together contradict our hypothesis $[X,K]_q \neq 0$.

If $[X,K]$ is proportional to K, then, because of their smoothness, a function $\widetilde{\Phi}$ exists on W for which $[X,K] = -\widetilde{\Phi}K$ and $\widetilde{\Phi}$ coincides with Φ on the orbit $O(p)$. Here X and K are Killing fields and K is null on W, thus $\widetilde{\Phi}K$ must be a null Killing vector field, therefore

$$0 = (\widetilde{\Phi}K_a)_{;b} + (\widetilde{\Phi}K_b)_{;a} = \widetilde{\Phi}_{;a}K_b + \widetilde{\Phi}_{;b}K_a .$$

Let $s \in W$ and $\{K, L, E_m\}$ be a pseudo-orthonormalized vector base at T_sM. Contracting the above equation with $K^a L^b$, $L^a L^b$, and $E^e_m L^b$ one obtains $\widetilde{\Phi}_{;a}K^a = 0$, $\widetilde{\Phi}_{;a}L^a = 0$, and $\widetilde{\Phi}_{;a}E^a_m = 0$, respectively; i.e., $d\widetilde{\Phi} = 0$ at s. But s can be chosen arbitrarily, therefore $\widetilde{\Phi} = \Phi_0 = \text{const}$ on W.

The orbit $O(p)$ is compact, so it can be covered by finitely many neighborhoods $W_1, \ldots, W_t$. But, due to the overlappings of the W's, Φ has to be the same constant value Φ_0 all along $O(p)$. This, however, implies $\Phi_0 = 0$, which contradicts the hypothesis $[X,K]_q \neq 0$. □

At the end of this section we review the properties of the two-dimensional orbits. If the action σ has a fixed point p, then a two-dimensional timelike submanifold, called the symmetry axis, can be foliated through p (Ref. 2), and the integral curve of $\widetilde{K}$ through p lies in this axis.

Outside the axis X and $\widetilde{K}$ together constitute a smooth two-dimensional involutive distribution,[7] thus there is a two-dimensional integral submanifold $N(p)$ of $X,\widetilde{K}$ through each nonfixed p. Here $N(p)$ is generated by the orbits $L(q)$ of the one-parameter group action generated by $\widetilde{K}$, and $q \in O(p)$. Since X and $\widetilde{K}$ are commuting Killing fields, all the inner products $g(X,X)$, $g(X,\widetilde{K})$, $g(\widetilde{K},\widetilde{K})$ are constant on $N(p)$. Therefore $N(p)$ is a cylinder with constant circumference and its causal character does not change along the orbits $L(q)$.

If $\widetilde{K}$ is null along $O(p)$, then K is null on $N(p)$ and the integral curves of K and $\widetilde{K}$, lying in $N(p)$, coincide. In this case, because of the Corollary of Proposition 3, and the equality $X(g(X,K)) = g(X,[X,K])$, $g(X,K)$ is constant on $N(p)$, too.

III. CYCLICALLY SYMMETRIC SPACE-TIMES WITH NULL KILLING SYMMETRY

If K is nonspacelike, then, in general, the causal character of $\widetilde{K}$ may vary on M, due to the changing of the causal character of K or of the commutation property of X and K.

Throughout this section K is assumed to be null. Therefore the first possibility above is ruled out, but, as our main theorem states, not the second one.

Theorem: Let (M,g) be cyclically symmetric with cyclic Killing vector field X and SO(2) action σ, and let K be a nowhere vanishing null Killing vector field on M. Then either $[X,K] = 0$ on M, or

$$U := \{\sigma(\varphi,p) \,|\, \varphi \in \mathrm{SO}(2),$$
$$p \in M: \nexists \alpha \in \mathrm{R} \text{ for which } [X,K]_p = \alpha K_p\}$$

is a nonempty open set and the Killing vector field $\widetilde{K}$ is timelike on U.

Proof: If $[X,K]$ is not zero at some point $p \in M$, then, as a corollary to Proposition 4, there is an open set V such that $p \in \overline{V}$ and $\widetilde{K}$ is timelike on V. But, as Proposition 3 states, $\widetilde{K}$ is timelike on U and can be timelike only on U. Here U is open and, because of $V \subseteq U$, nonempty. □

This theorem is the main result of the present paper. It states that the Killing vectors X and K can be noncommuting only if the space-time admits an additional timelike Killing symmetry on an open subset of M. Thus if we want to consider space-times only with cyclic and null Killing symmetries, we have to assume they commute, as otherwise stationarity on an open set is also assumed implicitly. (See Note added in proof.)

Recall that a null Killing vector field is always geodesic and its expansion and shear vanish. Thus space-times admitting a null Killing vector field K are classified as the twist ω of K vanishes or not, and in the first case as K is covariantly constant or not.[3] If, however, in addition a cyclic symmetry is also admitted, then a further subclass can be introduced.

Corollary: Let (M,g) be a cyclically and null Killing symmetric space-time with (commuting) Killing vector fields X and K, respectively, and SO(2) action σ.

(1) If $g(X,K) = 0$ throughout M then K is twist-free.

(2) If K is covariantly constant then $g(X,K)$ is constant on M, and, in addition, if σ has a fixed point then $g(X,K) = 0$.

Proof: (1) If p is not a fixed point of σ, then in a neighborhood W of p one can define a unit spacelike smooth vector field Y, being orthogonal to both K and X. This Y is unique up to a sign. The function $x := (g(X,X))^{1/2}$ is nonzero and smooth on W, thus $E_2 := Y$, $E_3 := (1/x)X$ constitute a smooth two-dimensional orthonormal spacelike base field on W, being orthogonal to K. The twist of K can be calculated in this base,

$$2\omega x = xg(\nabla_{E_2}K,E_3) - xg(\nabla_{E_3}K,E_2)$$
$$= -g(K,\nabla_Y X) + g(K,\nabla_X Y) = g(K,[X,Y]) ;$$

i.e., if Y is Lie propagated along X then K is twist-free.

Let $q \in W$ and let Y' denote the vector field along the orbit $O(p)$, obtained by Lie propagation of the vector Y_q. [Here X is a Killing field, thus, in spite of the fact that $O(p)$ is closed,

Y' is well defined all along $O(p)$.] Then

$$X(g(X,Y')) = Y'^a X_{a;b} X^b + X^a Y'_{a;b} X^b$$
$$= X^a (Y'_{a;b} X^b - X_{a;b} Y'^b) = 0\,,$$
$$X(g(K,Y')) = Y'^a K_{a;b} X^b + K^a Y'_{a;b} X^b$$
$$= Y'^a X_{a;b} K^b + K^a Y'_{a;b} X^b$$
$$= K^a (Y'_{a;b} X^b - X_{a;b} Y'^b) = 0\,,$$
$$\tfrac{1}{2} X(g(Y',Y')) = Y'^a Y'_{a;b} X^b = Y'^a X_{a;b} Y'^b = 0\,;$$

i.e., Y' is the unit spacelike vector field being orthogonal to X and K all along $O(p)$. Thus it coincides with Y; i.e., Y is Lie propagated along X.

(2) Let q be a nonfixed point and let I_q denote the integral of $g(X,K)$ on $O(p)$. As we stated at the end of the previous section, $g(X,K)$ is constant along the orbits $N(q)$, thus $I_q = 2\pi g_q(X,K)$. On the other hand, I_q can be considered as the integral of the closed one-form field K_a on the one-cycle $O(q)$ (K_a is closed; i.e., $K_{[a;b]} = 0$, because K_a is constant):

$$I_q = \int_{O(q)} K\,.$$

Let q' be an arbitrary point of M. Here M is connected, therefore there is a smooth curve μ: $[0,1] \to M$ from $q = \mu(0)$ to $q' = \mu(1)$. The mapping

$$F\colon [0,1] \times [0,2\pi] \to M\colon (t,\psi) \mapsto \sigma(\psi,\mu(t))$$

is a smooth homotopy between the orbits $O(q)$ and $O(q')$, thus $I_q = I_{q'}$, which implies $g(X,K) = \text{const}$. If there is a fixed point p, then q' can be chosen to be p, therefore, because of $O(p) = \{p\}$, every orbit $O(q)$ is homotopic to zero and consequently $g(X,K) = 0$. □

This Corollary gives a sufficient condition on a cyclically and null Killing symmetric space-time to be in Kundt's class,[3] moreover it states that in physically important axisymmetric space-times describing *pp* waves[3] the Killing vector K must be orthogonal to the orbits of axisymmetry.

Finally, it is worth noting that the solution of Lukács, Perjés, and Sebestyén[4] has a twisting null Killing vector and the second Killing vector is a cyclic one on an open domain. These vectors commute and have nonzero inner product, as it must be according to our Theorem and its Corollary.

Note added in proof: For the sake of completeness it should be noted that the additional timelike Killing vector $\widetilde{K}$ is independent of K and X on U; i.e., there are not functions α and β on U for which $\widetilde{K} = \alpha K + \beta X$ would hold.

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