Class. Quantum Grav. 10 (1993) 1899–1905. Printed in the UK

On the positivity of the quasi-local mass

L B Szabados†

Central Research Institute for Physics, H-1525 Budapest 114, POBox 49, Hungary

Received 17 February 1993

Abstract. The quasi-local energy-momentum of Dougan and Mason, associated with a spacelike topological two-sphere, is examined. It is shown that the energy-momentum four-vector is zero iff the Cauchy development of the 3-surfaces spanned by the two-sphere is flat, and is null (i.e. the quasi-local mass is zero) if and only if the Cauchy development is a pure radiative *pp*-wave spacetime geometry with common principle null direction of the Weyl and Ricci tensors.

PACS number: 0420

1. Introduction

One of the most important results of classical general relativity during the last decade is the proof of positivity of the gravitational energy both at null and spatial infinity. More precisely, it was shown [1] that the energy–momentum four-vector, associated with a spacelike hypersurface Σ extending either to null or spacelike infinity, is future-directed and non-spacelike, provided the matter fields satisfy the dominant energy condition. It has also been demonstrated [2] that the energy–momentum is *strictly* timelike and is null iff the Cauchy development of Σ is flat.

Recently Dougan and Mason proposed a new definition for the energy–momentum and mass in general relativity at the *quasi-local* level [3]. This energy–momentum, associated with a spacelike topological two-sphere $\mathcal{S}$, has several satisfactory properties. It gives (1) zero in flat spacetime, (2) the correct results in linearized theory, for round spheres and for small spheres; and (3) the ADM and Bondi–Sachs masses at spacelike and future null infinity, respectively. Finally (4) it was also proved that the quasi-local energy, a component of the quasi-local four-momentum, is non-negative if the dominant energy condition is satisfied on the hypersurfaces Σ spanned by $\mathcal{S}$ and $\mathcal{S}$ is 'concave'. This, however, does not imply that the gravitational energy–momentum is non-zero for non-flat Cauchy developments of Σ, and would be useful to see whether the quasi-local mass may be null for certain non-flat Cauchy developments.

In the present paper we complete the positivity analysis of Dougan and Mason showing that, under the same conditions, the vanishing of the energy–momentum four-vector implies the flatness of the Cauchy development and the gravitational quasi-local mass is null iff the Cauchy development admits a covariantly constant null Killing vector (i.e. it is a gravitational *pp*-wave), the matter is pure radiation and their wavefronts coincide.

† E-mail address: lbszab@rmk520.rmki.kfki.hu

0264-9381/93/091899+07$07.50

2. The quasi-local integrals

Let \$ be a two-dimensional spacelike closed oriented submanifold of M, and for any pair of spinor fields λ_R and μ_S define

$$I_\$[\lambda,\bar{\mu}] := \frac{2}{k}\oint_\$ \mathrm{i}\bar{\mu}_{A'}\nabla_{BB'}\lambda_A \mathrm{d}x^a \wedge \mathrm{d}x^b. \tag{1}$$

(The signature of the metric is $(+---)$, the curvature and Ricci tensors are $R^a{}_{bcd}X^b := -(\nabla_c\nabla_d - \nabla_d\nabla_c)X^a$ and $R_{bd} := R^a{}_{bad}$, respectively. Einstein's equations are written in the form $G_{ab} := R_{ab} - \frac{1}{2}Rg_{ab} = -kT_{ab}$. Throughout this paper the abstract index formalism will be used [4] and underlined indices have numerical values.) $I_\$[\lambda,\bar{\mu}]$, the integral of the so-called Nester–Witten form [5], is easily seen to be a Hermitian bilinear functional on the space $C^\infty(\$, \mathbf{S}_A)$ of smooth spinor fields on \$. The importance of the quasi-local integral (1) is shown by the Sparling equation [5]:

$$\begin{aligned}\mathrm{d}\Big(i\bar{\mu}_{A'}\nabla_{BB'}\lambda_A \mathrm{d}x^a \wedge \mathrm{d}x^b\Big) &= \mathrm{i}\nabla_{CC'}\bar{\mu}_{A'}\ nabla_{BB'}\lambda_A \mathrm{d}x^a \wedge \mathrm{d}x^b \wedge \mathrm{d}x^c \\ &\quad - \tfrac{1}{2}\lambda^A\bar{\mu}^{A'}G_a{}^b\frac{1}{3!}\varepsilon_{bcde}\mathrm{d}x^c \wedge \mathrm{d}x^d \wedge \mathrm{d}x^e.\end{aligned} \tag{2}$$

The first term on the right, denoted by $\Gamma(\lambda,\bar{\mu})$, is the so-called Sparling form. It is 'Hermitian' in the sense that $\Gamma(\lambda,\bar{\mu}) = \overline{\Gamma(\mu,\bar{\lambda})}$. If Σ is a three-dimensional spacelike hypersurface with boundary \$ then the pullback of (2) along the natural imbedding $i : \Sigma \to M$ is equivalent to the Reula–Tod form [1] of the Sen–Witten identity. Thus $I_\$[\lambda,\bar{\mu}]$ is related to the energy–momentum of the gravitating system surrounded by \$. One has, however, to specify the 'propagation law(s)' for the spinor fields λ_R and μ_S within \$.

3. The 'propagation laws'

It is natural to look for the 'propagation law' for the spinor fields in the form $\not{D}\lambda_R = 0$, where $\not{D}$ is an elliptic first-order differential operator on $C^\infty(\$, \mathbf{S}_A)$. Dougan and Mason considered anti-holomorphic (holomorphic) spinor fields, i.e. those satisfying $m^a\nabla_a\lambda_R = 0$ ($\bar{m}^a\nabla_a\lambda_R = 0$). In the GHP formalism [4,6], where the normalized spinor dyad (o_A, ι_A) is chosen so that $m^a = o^A\bar{\iota}^{A'}$ is the usual complex null vector tangent to \$, the condition of anti-holomorphity and holomorphity is equivalent to

$$-\Delta^-\lambda := \eth\lambda_1 + \rho'\lambda_0 = 0 \qquad \mathcal{T}^-\lambda := \eth\lambda_0 + \sigma\lambda_1 = 0; \tag{3}$$

and

$$\Delta^+\lambda := \eth'\lambda_0 + \rho\lambda_1 = 0 \qquad -\mathcal{T}^+\lambda := \eth'\lambda_1 + \sigma'\lambda_0 = 0 \tag{4}$$

respectively.

One can develop [7] a covariant two-dimensional spinor calculus being the two-dimensional version of the usual Sen connection [8]. $\Delta^\pm\lambda$ and $\mathcal{T}^\pm\lambda$ are the GHP form of the irreducible chiral parts of the (two-dimensional) Sen-derivative of the spinor field λ_R in that formalism. $\Delta^+ \oplus \Delta^-$ is the GHP form of the (two-dimensional) Weyl–Sen–Witten, while $\mathcal{T}^+ \oplus \mathcal{T}^-$ is of the (two-dimensional) twistor operators. $\mathcal{H}^\pm := \Delta^\pm \oplus \mathcal{T}^\pm$ are elliptic first-order differential operators and index $\mathcal{H}^\pm = 2(1-\mathcal{G})$, where $\mathcal{G}$ is the genus of \$. Thus for topological two-spheres there are *at least* two holomorphic and two anti-holomorphic spinor fields on \$. If $\lambda^{\underline{A}}_R$, $\underline{A} = 0, 1$ are two linearly independent anti-holomorphic spinor fields, then $m^a \nabla_a (\varepsilon^{RS} \lambda^{\underline{A}}_R \lambda^{\underline{B}}_S) = 0$ and by Liouville's theorem $\varepsilon^{RS}\lambda^{\underline{A}}_R\lambda^{\underline{B}}_S$ is constant on \$. If this constant is not zero then it can be chosen to be $\epsilon^{\underline{A}\underline{B}}$, the Levi-Cività alternating symbol, and $\{\lambda^{\underline{A}}_R\}$ form a normalized spinor basis at each point of \$. Then for any spinor field λ_R on \$ there are complex functions α and β such that $\lambda_R = \alpha\lambda^0_R + \beta\lambda^1_R$. If λ_R is anti-holomorphic then α and β are anti-holomorphic complex functions on \$, and hence by Liouville's theorem they must be constant. Thus in this case the space of anti-holomorphic spinor fields on a spherical \$ is two-complex-dimensional. If however $\lambda^0_R \lambda^1_S \varepsilon^{RS} = 0$ then both $\lambda^{\underline{A}}_R$ must have a zero and $\lambda^1_R = f\lambda^0_R$ for some anti-meromorphic function f on \$ with a pole at a zero of λ^0_R, and hence λ^0_R and λ^1_R do not form a basis in the spinor spaces. (I am grateful to one of the referees of the previous version of this paper for clarifying this second possibility.) As is usual, we will assume that the geometry of \$ is not exceptional in the sense that there are precisely two linearly independent anti-holomorphic spinor fields spanning the spinor spaces at each point of \$ [3]. Other propagation laws were proposed by Bergqvist [9].

4. The non-negativity of the quasi-local mass

The GHP form of (1) is

$$I_\$[\lambda, \bar{\mu}] = \frac{2}{k} \oint_\$ \left\{ \bar{\mu}_{1'}\left(\eth'\lambda_0 + \rho\lambda_1\right) - \bar{\mu}_{0'}\left(\eth\lambda_1 + \rho'\lambda_0\right) \right\} \mathrm{d}\$, \tag{5}$$

and hence $\ker(\Delta^+ \oplus \Delta^-) = \ker I_\$$. Obviously $\ker\Delta^-$ is an infinite-dimensional subspace of $C^\infty(\$, \mathbf{S}_A)$ and $I_\$$ is a Hermitian bilinear functional on $\ker\Delta^-$ too; and consider only spinor fields λ_R belonging to $\ker\Delta^-$. Suppose that \$ is the boundary of some spacelike hypersurface Σ with unit future normal t^a. $h_{ab} := g_{ab} - t_a t_b$ is the induced negative definite metric on Σ and $\mathcal{D}_e := h_e{}^f \nabla_f$ is the (three-dimensional) Sen operator [1, 8]. If $\tilde{\lambda}_R$ is a spinor field on Σ satisfying the Sen–Witten equation $\mathcal{D}_{R'}{}^R\tilde{\lambda}_R = 0$ with boundary condition $\tilde{\lambda}_1|_\$ = \lambda_1$ (such a $\tilde{\lambda}_R$ always exists) then one has [3] (see also [9])

$$I_\$[\lambda, \bar{\lambda}] = \frac{2}{k}\oint_\$ \rho' |\lambda_0 - \tilde{\lambda}_0|^2 \mathrm{d}\$ + \\ + \int_\Sigma \left\{ -\frac{2}{k} h^{ef} t^{AA'} \left(\mathcal{D}_e\tilde{\lambda}_A\right)\left(\mathcal{D}_f\bar{\tilde{\lambda}}_{A'}\right) + \tilde{\lambda}^A \bar{\tilde{\lambda}}^{A'} T_{ab} t^b \right\} \mathrm{d}\Sigma. \tag{6}$$

Thus if $\rho' \geqslant 0$ on \$ and if the dominant energy condition holds on Σ then the bilinear form $I_\$$ is non-negative on $\ker\Delta^-$. It therefore satisfies the Cauchy–Schwartz inequality

$$I_\$[\lambda, \bar{\lambda}] I_\$[\mu, \bar{\mu}] \geqslant I_\$[\lambda, \bar{\mu}] I_\$[\mu, \bar{\lambda}] \tag{7}$$

for any spinor fields $\lambda_R, \mu_S \in \ker\Delta^-$.

Let $\lambda^{\underline{A}}_R$, $\underline{A} = 0, 1$ be a normalized basis in the space of anti-holomorphic spinor fields. Then the Dougan–Mason quasi-local four-momentum, energy and mass [3] are

$$P^{\underline{A}\underline{B}'}_{\$} := I_{\$}[\lambda^{\underline{A}}, \bar{\lambda}^{\underline{B}'}] \tag{8}$$

$$E_{\$} := \frac{1}{\sqrt{2}}\left(P^{00'}_{\$} + P^{11'}_{\$}\right) \tag{9}$$

$$m^2_{\$} := \epsilon_{\underline{A}\underline{B}}\epsilon_{\underline{A}'\underline{B}'}P^{\underline{A}\underline{A}'}_{\$}P^{\underline{B}\underline{B}'}_{\$} = 2\left(P^{00'}_{\$}P^{11'}_{\$} - P^{01'}_{\$}P^{10'}_{\$}\right) \tag{10}$$

respectively. Thus $\rho' \geqslant 0$ and the dominant energy condition on Σ ensure the non-negativity of $E_{\$}$ and, by (7), of $m^2_{\$}$; i.e. the quasi-local energy–momentum is a future-directed non-spacelike vector. In the rest of this paper $\rho' \geqslant 0$ and the dominant energy condition will be assumed to hold.

5. Zero energy implies flatness

Since the quasi-local energy–momentum is zero in flat spacetime it is natural to ask whether its converse is true or not; i.e. whether there might be non-flat spacetime regimes with vanishing energy–momentum vector. Because of the Cauchy–Schwartz inequality and the non-negativity of $I_{\$}$, $P^{\underline{A}\underline{A}'}_{\$} = 0$ is equivalent to $E_{\$} = 0$; i.e. to $I_{\$}[\lambda^0, \bar{\lambda}^{0'}] = I_{\$}[\lambda^1, \bar{\lambda}^{1'}] = 0$. Then, because of the positive definiteness of $-h^{ef}$ and $t^{AA'}$, from equation (6) $\mathcal{D}_e\bar{\lambda}^0_R = \mathcal{D}_e\bar{\lambda}^1_R = 0$, and hence $\bar{\lambda}^{0A}R_{ABef}h^e{}_ch^f{}_d = \bar{\lambda}^{1A}R_{ABef}h^e{}_ch^f{}_d = 0$; i.e. $R_{ABef}h^e{}_ch^f{}_d = 0$ follow. Since Σ is arbitrary, this must hold for any deformation of Σ, and hence $R_{ABcd} = 0$. Thus the Cauchy development $D(\Sigma)$ of Σ is flat (see also [2, 10]).

6. Zero-mass Cauchy developments

Although the *total* gravitational energy–momentum is strictly timelike [2, 10], neither the energy-non-negativity proof nor the previous argumentation imply the *positivity* of the quasi-local mass; i.e. that the gravitational quasi-local energy–momentum is *strictly* timelike. In fact, we show that the vanishing of $m^2_{\$}$ does not imply flatness, and the non-flat spacetime geometries with vanishing mass are precisely the *pp*-wave spacetimes with pure radiation having common principal null direction.

If $m^2_{\$}$ is zero, then either at least one of $P^{00'}_{\$}$ and $P^{11'}_{\$}$ is zero or both are non-zero but the equality holds in the Cauchy–Schwartz inequality. If both $P^{00'}_{\$}$ and $P^{11'}_{\$}$ were non-zero then $I_{\$}$ would be a positive definite Hermitian inner product on the two-dimensional complex vector space spanned by λ^0_R and λ^1_R. But then the equality in the Cauchy–Schwartz inequality would imply the proportionality of λ^0_R and λ^1_R. Thus $I_{\$}[\lambda, \bar{\lambda}] = 0$, where λ_R is either λ^0_R or λ^1_R. Since $I_{\$}[\lambda, \bar{\lambda}] = 0$ only for one independent spinor field, one should modify the argumentation of the previous section to get restrictions for the curvature from $I_{\$}[\lambda, \bar{\lambda}] = 0$. First we show that at the points of $D(\Sigma)$ the curvature is zero or of Petrov-N type with pure radiation [11].

Let p be a point of the interior of the Cauchy development of Σ. One can always assume $p \in \Sigma - \partial\Sigma$. Let $\{t^a, E^a_{\underline{i}}\}$, $\underline{i} = 1, 2, 3$, be an orthonormal basis at T_pM with t^a the future-directed timelike normal to Σ. Let $\Sigma_{\underline{i}}(u)$ be 1-parameter deformations of Σ around p with normal $t^a \cosh u + E^a_{\underline{i}} \sinh u$ at $p \in \Sigma \cap \Sigma_{\underline{i}}(u)$ and $h^e_{\underline{i}f}(u)$ the corresponding projections.

If $\tilde{\lambda}^A_{\underline{i}}(u)$ are the solutions of the Sen–Witten equation on $\Sigma_{\underline{i}}(u)$ with the fixed boundary condition $\lambda_1 = \tilde{\lambda}_{\underline{i}\,1}(u)|_{\$}$ then by equation (6) from $I_{\$}[\lambda, \bar{\lambda}] = 0$ $h^f_{\underline{i}\ e}(u)\nabla_f\tilde{\lambda}_{\underline{i}\,R}(u) = 0$ and, as above, $\tilde{\lambda}^A_{\underline{i}}(u)R_{ABef}h^e_{\underline{i}\,c}(u)h^f_{\underline{i}\ d}(u) = 0$ follow. Suppose that $\dot{\tilde{\lambda}}^R_{\underline{i}}$ is not proportional to $\tilde{\lambda}^R$ at p for some $\underline{i}$, say $\underline{i} = 3$, where the dot denotes the differentiation with respect to u at $u = 0$. Then $\dot{\tilde{\lambda}}^R_3$ and $\tilde{\lambda}^R$ form a basis in the spinor space at p, and hence $\dot{\tilde{\lambda}}^R_1 = \alpha\dot{\tilde{\lambda}}^R_3 + \gamma\tilde{\lambda}^R$ and $\dot{\tilde{\lambda}}^R_2 = \beta\dot{\tilde{\lambda}}^R_3 + \delta\tilde{\lambda}^R$ for some $\alpha, \beta, \gamma, \delta \in \mathbf{C}$. Then for the curvature at p we have

$$0 = \alpha\dot{\tilde{\lambda}}^A_3 R_{ABef}h^e{}_c h^f{}_d + \tilde{\lambda}^A R_{ABef}\left(h^e{}_c\dot{h}^f_{1\,d} + \dot{h}^e_{1c}h^f{}_d\right)$$

$$0 = \beta\dot{\tilde{\lambda}}^A_3 R_{ABef}h^e{}_c h^f{}_d + \tilde{\lambda}^A R_{ABef}\left(h^e{}_c\dot{h}^f_{2\,d} + \dot{h}^e_{2c}h^f{}_d\right)$$

$$0 = \dot{\tilde{\lambda}}^A_3 R_{ABef}h^e{}_c h^f{}_d + \tilde{\lambda}^A R_{ABef}\left(h^e{}_c\dot{h}^f_{3\,d} + \dot{h}^e_{3c}h^f{}_d\right)$$

$$0 = \tilde{\lambda}^A R_{ABef}h^e{}_c h^f{}_d.$$

Now it is a (rather long but) direct calculation to show that this system of equations has non-trivial independent solutions for $\dot{\tilde{\lambda}}^A_3$ and $\tilde{\lambda}^A$ only if $R_{ABcd} = 0$. If $\dot{\tilde{\lambda}}^A_{\underline{i}}$ is proportional to $\tilde{\lambda}^A$ at p for all $\underline{i} = 1, 2, 3$ then $\tilde{\lambda}^A R_{ABef}h^e_{\underline{i}\,c}(u)h^f_{\underline{i}\ d}(u) = 0$ and hence $\tilde{\lambda}^A R_{ABcd} = 0$. This, together with the dominant energy condition, is equivalent to

$$\psi_{ABCD} = \psi\tilde{\lambda}_A\tilde{\lambda}_B\tilde{\lambda}_C\tilde{\lambda}_D \quad \phi_{ABA'B'} = \tfrac{1}{2}\phi\tilde{\lambda}_A\tilde{\lambda}_B\bar{\tilde{\lambda}}_{A'}\bar{\tilde{\lambda}}_{B'} \quad \Lambda = 0 \tag{11}$$

where ψ is complex and ϕ is real and non-negative. The third condition $\tilde{\lambda}^A\bar{\tilde{\lambda}}^{A'}G_{ab}t^b = 0$, coming from $I_{\$}[\lambda, \bar{\lambda}] = 0$ and (6), is automatically satisfied. If ρ' is not identically zero on \$ then by the first condition $\rho'|\lambda_0 - \tilde{\lambda}_0|^2 = 0$, coming from $I_{\$}[\lambda, \bar{\lambda}] = 0$ and (6), λ_R and $\tilde{\lambda}_R$ coincide on supp $\rho' \subset \$$; and hence by Liouville's theorem on the whole \$. Two independent solutions to $\mathcal{D}_e\tilde{\lambda}_R = 0$ with the given boundary condition can exist only if $\rho' = 0$ on \$ and, following the argumentation of the previous section, $D(\Sigma)$ is flat. If therefore we fix a smooth one-parameter foliation Σ_t of $D(\Sigma)$ such that $\partial\Sigma_t = \$$ then we have a smooth spinor field $\tilde{\lambda}_R$ on $D(\Sigma)$ which is constant on each Σ_t, coincides with λ_R on \$ and is a repeated principle spinor of the curvature. We will show that this spinor field is constant on $D(\Sigma)$ and hence $\tilde{L}^a := \tilde{\lambda}^A\bar{\tilde{\lambda}}^{A'}$ is a constant null vector.

Let the basis field $\{E^a_{\underline{i}}\}$ be chosen so that $\chi\bar{\chi}\sqrt{2}\tilde{L}^a = t^a + E^a_3$ for some non-zero complex function χ. To prove that $\tilde{\lambda}_R$ is constant we introduce a normalized spinor dyad (O_A, I_A) and, for the sake of convenience, use the GHP formalism [6] with the well known notations for the operators, ð and Þ, and the spin coefficients. These, however, should not be confused with those introduced on the two-sphere \$.

Let $O_A := \chi\tilde{\lambda}_A$ and I_A be such that $O_AI^A = 1$ and for the vectors of the complex null tetrad $L^a := O^A\bar{O}^{A'}$, $M^a := O^A\bar{I}^{A'}$, $\bar{M}^a := \bar{O}^{A'}I^A$ and $N^a := I^A\bar{I}^{A'}$ we require that $L^a + N^a = \sqrt{2}t^a$, $L^a - N^a = \sqrt{2}E^a_3$ and that M^a, $\bar{M}^a$ be tangents to Σ_t. For fixed χ these conditions fix the spinor field I_A completely. Note that χ is of $(1, 0)$ type. With this choice of the dyad the condition of hypersurface orthogonality of t^a is equivalent to $\rho - \bar{\rho} - \rho' + \bar{\rho}' = 0$ and $\kappa - \bar{\kappa}' - \tau + \bar{\tau}' + 2(\beta - \bar{\beta}') = 0$. The condition $\mathcal{D}_a\tilde{\lambda}_R = 0$ can thus be rewritten as $L^a\nabla_a\tilde{\lambda}_R = N^a\nabla_a\tilde{\lambda}_R = O_R\text{Þ}(\frac{1}{\chi}) - I_R\frac{\kappa}{\chi}$, $M^a\nabla_a\tilde{\lambda}_R = 0$, $\bar{M}^a\nabla_a\tilde{\lambda}_R = 0$; and hence $\nabla_a\tilde{L}_b = (L_a + N_a)L_b\text{Þ}(\frac{1}{\chi\bar{\chi}}) - \frac{1}{\chi\bar{\chi}}(L_a + N_a)(M_b\bar{\kappa} + \bar{M}_b\kappa)$. In terms of spin coefficients this is equivalent to $\sigma = \rho = \kappa - \tau = 0$, $\beta = \frac{1}{\chi}\delta\chi$, $\beta' = -\frac{1}{\chi}\bar{\delta}\chi$ and $\varepsilon + \varepsilon' = \frac{1}{\chi}(D\chi - \Delta\chi)$.

The last three can be rewritten as $\eth\chi = \eth'\chi = (\text{Þ} - \text{Þ}')\chi = 0$. Recalling that the only non-zero curvature components are ψ_4 and ϕ_{22}, and using the above restrictions on the spin coefficients from the spinor Bianchi identities we have $\kappa\psi_4 = 0$ and $\kappa\phi_{22} = 0$. For non-flat spacetime, κ must therefore be zero. Thus $\tilde{L}^a$ is autoparallel, and we will show that $\text{Þ}\chi$ also vanishes.

The non-trivial commutators of the edth and thorn operators are $\text{ÞÞ}' - \text{Þ}'\text{Þ} = -\tau'\eth - \bar{\tau}'\eth'$, $\text{Þ}\eth - \eth\text{Þ} = -\bar{\tau}'\text{Þ}$ and $\text{Þ}'\eth - \eth\text{Þ}' = \rho'\eth + \bar{\sigma}'\eth' - \bar{\kappa}'\text{Þ}$. Applying the first commutator to χ and using $\eth\chi = \eth'\chi = (\text{Þ} - \text{Þ}')\chi = 0$ we get

$$0 = (\text{Þ} - \text{Þ}')\text{Þ}\chi = (D - \Delta - 2\varepsilon - \bar{\varepsilon} - 2\varepsilon' - \bar{\varepsilon}')\text{Þ}\chi$$
$$0 = (\text{Þ} - \text{Þ}')\text{Þ}'\chi = (D - \Delta + \bar{\varepsilon} + \bar{\varepsilon}')\text{Þ}'\chi$$

and hence $(\varepsilon+\bar{\varepsilon}+\varepsilon'+\bar{\varepsilon}')\text{Þ}\chi = 0$. Thus $\text{Þ}\chi = 0$ or $0 = \varepsilon+\bar{\varepsilon}+\varepsilon'+\bar{\varepsilon}' = \frac{1}{\chi\bar{\chi}}(D(\chi\bar{\chi}) - \Delta(\chi\bar{\chi}))$; i.e. $E^a_3\nabla_a(\chi\bar{\chi}) = 0$. From the difference of the second and third commutators we have $0 = (\text{Þ} - \text{Þ}')\eth\chi - \eth(\text{Þ} - \text{Þ}')\chi = (\bar{\kappa}' - \bar{\tau}')\text{Þ}\chi$. Thus $\text{Þ}\chi = 0$ or $0 = \bar{\kappa}' - \bar{\tau}' = \frac{2}{\chi\bar{\chi}}\delta(\chi\bar{\chi})$; i.e. $E^a_1\nabla_a(\chi\bar{\chi}) = E^a_2\nabla_a(\chi\bar{\chi}) = 0$. Thus $\text{Þ}\chi = 0$ or $\chi\bar{\chi}$ is constant on each Σ_t. Because of the definition of χ, however, $\mathcal{D}_e t_f E^f_3 = -\frac{1}{\chi\bar{\chi}}\mathcal{D}_e(\chi\bar{\chi})$, thus for constant $\chi\bar{\chi}$ E^f_3 would have to be an eigenvector of the extrinsic curvature $\mathcal{D}_e t_f$ of Σ_t with zero eigenvalue. For general foliation, however, there are no such eigenvectors. The Cauchy development $D(\Sigma)$ is therefore a *pp*-wave geometry with pure radiation.

Conversely, suppose that there is a constant null vector field $L^a = \lambda^A\bar{\lambda}^{A'}$ on $D(\Sigma)$. Then, with an appropriate choice of the phase $e^{i\alpha}$, $\tilde{\lambda}^A := e^{i\alpha}\lambda^A$ is constant on $D(\Sigma)$. Thus it is constant on $\$$ too; and hence, by (3) and (4), it is holomorphic *and* anti-holomorphic. $\tilde{\lambda}_R$ can therefore be used as one of the basis spinors and by $\tilde{\lambda}_R \in \ker(\Delta^+ \oplus \Delta^-)$ the quasi-local integral $I_\$[\tilde{\lambda}, \bar{\tilde{\lambda}}]$, and hence the quasi-local mass is zero.

7. Discussion

There is a more or less commonly accepted list of criteria of resonableness of the quasi-local energy–momentum expressions (see introduction and e.g. [12]). This includes (1) the requirement of the vanishing of the quasi-local energy–momentum for flat spacetimes; and (4) the non-negativity of the quasi-local energy. (1), however, does not exclude the possibility of zero energy–momentum for non-flat spacetimes, which would be desirable from physical points of view. Thus it seems reasonable to strenghten (1) by requiring the vanishing of the quasi-local four-momentum *precisely* for flat spacetime geometries. To retain the four-dimensional character of the criteria of reasonableness, (4) should obviously be reformulated by requiring the quasi-local four-momentum be a future directed non-spacelike vector. Thus one can ask whether this four-momentum is strictly timelike or may be null.

Suppose that M is the Minkowski spacetime, Σ is a spacelike hypersurface with boundary $\$ = \partial\Sigma$ homeomorphic to S^2; and consider the integral ${}_M I_\$[\lambda, \bar{\mu}] := \int_\Sigma \lambda_A\bar{\mu}_{A'}T^{ab}\mathrm{d}\Sigma_b$, where λ_A, μ_A are *constant* spinor fields on M. ${}_M I_\$$ is a well defined Hermitian bilinear functional on the space $\mathbf{S}_A(M)$ of constant spinor fields on M. If T_{ab} satisfies the dominant energy condition then ${}_M I_\$$ is non-negative and hence satisfies the Cauchy–Schwartz inequality. Then the quasi-local four-momentum and mass of the matter fields associated with $\$$ are defined by ${}_M P^{\underline{A}\underline{B}'}_\$:= {}_M I_\$[\lambda^{\underline{A}}, \bar{\lambda}^{\underline{B}'}]$ and ${}_M m^2_\$:= \epsilon_{\underline{A}\underline{B}}\epsilon_{\underline{A}'\underline{B}'}{}_M P^{\underline{A}\underline{A}'}_\$ {}_M P^{\underline{B}\underline{B}'}_\$$, respectively, where $\lambda^0_R, \lambda^1_R \in \mathbf{S}_A(M)$ such that $\lambda^0_R\lambda^1_S\varepsilon^{RS} = 1$. The dominant energy condition ensures that ${}_M P_{\$}^{\underline{A}\underline{A}'}$ is a future directed non-spacelike vector. It is easy to see that ${}_M P_{\$}^{\underline{A}\underline{A}'} = 0$ iff T_{ab} is zero on $D(\Sigma)$, the Cauchy development of Σ; and ${}_M m_{\$}^2 = 0$ iff $\lambda^A \bar{\lambda}^{A'} T_{ab} = 0$ on $D(\Sigma)$ for some constant spinor field λ_A, i.e. the matter is a pure radiation on $D(\Sigma)$. Thus the vanishing of the gravitational quasi-local mass may be expected to characterize the pure radiative spacetime geometries and this requirement may be added to the list of the criteria of the reasonableness of the gravitational quasi-local energy–momentum expressions.

The present calculations can be considered as additional tests of the energy–momentum expression of Dougan and Mason. The results are in accordance with our physical picture: the Dougan–Mason energy–momentum satisfies the strengthen criterion (1) above, and the vanishing of the quasi-local mass is equivalent to a pure radiative spacetime geometry. On the other hand, having accepted this expression as the 'correct' gravitational energy–momentum, the results can also be interpreted as the determination/definition of the 'elementary states' of classical gravity: the minimal (zero) energy state is just the 'ground state' defined by the vanishing of the field strengths and particle fields; and the non-trivial zero-mass states are all plane waves.

References

[1] Reula O, Tod K P 1984 *J. Math. Phys.* **25** 1004

[2] Ashtekar A, Horowitz G T 1982 *Phys. Lett.* **89** A 181

[3] Dougan A J, Mason L J 1991 *Phys. Rev. Lett.* **67** 2119

[4] Penrose R, Rindler W 1982 *Spinors and Spacetime* vol 1 (Cambridge: Cambridge University Press)

[5] Mason L J, Frauendiener J 1990 *Twistors in Mathematics and Physics* (London Math. Soc. Lecture Note Series) ed R Baston and T Bailey (Cambridge: Cambridge University Press)

[6] Geroch R, Held A, Penrose R 1973 *J. Math. Phys.* **14** 874

[7] Szabados L B 1993 Two-dimensional Sen connections *Proc. 4th Hungarian Relativity Workshop, July 12-17, 1992, Gárdony* ed R P Kerr and Z Perjés (to be published); Two dimensional Sen connection and quasi-local energy–momentum and angular momentum (in preparation)

[8] Sen A 1981 *J. Math. Phys.* **22** 1781

[9] Bergqvist G 1992 *Class. Quantum Grav.* **9** 1917

[10] Horowitz G T 1984 The positive energy theorem and its extensions *Asymptotic Behavior of Mass and Spacetime Geometry, Lecture Notes in Phys. no 202* ed F J Flaherty (New York: Springer)

[11] Kramer D *et al* 1980 *Exact Solutions of Einstein's Field Equations* (Berlin: VEB Deutscher Verlag der Wissenschaften)

[12] Christodoulou D and Yau S-T 1988 Some remarks on the quasi-local mass *Mathematics and General Relativity* (Contemporary Mathematics 71) ed J A Isenberg (New York: AMS)