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On canonical pseudotensors, Sparling's form and Noether currents

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Abstract. This paper describes a differential geometric unification and reformulation of earlier pseudotensorial approaches. It is shown that, along coordinate sections, the pull backs of the contravariant and dual forms of Sparling's form, defined on the bundle of linear frames $L(M)$ over the m-dimensional spacetime M, are the Bergmann and the Landau–Lifshitz pseudotensors, respectively. Although the pull backs of Sparling's form along rigid sections are not exactly the energy–momentum tensors of the rigid basis description of gravity, they are always tensorial and the pull backs of the full Sparling equation are always the equations expressing the canonical (pseudo) tensors by the corresponding superpotentials. For any vector field on the spacetime an $(m-1)$ form, called the Noether form, is defined on $L(M)$ whose pull backs to the spacetime are, however, always the corresponding canonical Noether (pseudo) currents. It is shown that the Noether form is just the canonical Noether current, associated with the horizontal lift of a vector field on M, in the sense of a Lagrangian scenario on the bundle $L(M)$. For certain non-horizontal lifts the canonical Noether current is the sum of the Noether form and Komar's tensorial current. A $gl(m,\boldsymbol{R})$ valued $(m-1)$ form, called the spin form, is defined on $L(M)$, and it is shown that its pull backs are the various canonical spin (pseudo) tensors. In terms of the spin form and an energy–momentum m form a necessary and sufficient condition is found for the metric connection to be torsion free and to satisfy Einstein's equations. An exterior differential equation for the contravariant form of the spin and energy–momentum forms is derived, the pull backs of this equation are just the Belinfante–Rosenfeld equations for the canonical (pseudo) tensors. From the reformulation the possibility arises of completing the Landau–Lifshitz pseudotensor by adding a spin term. However, in the Landau–Lifshitz approach the orbital and spin angular momenta are separately conserved, from which it follows that the Landau–Lifshitz pseudotensors are probably not physical.

1. Introduction

In the classical Lagrangian field theory there are two sharply different classes of fields on the base manifold M: the matter fields and the fields specifying the spacetime geometry. The latter consists of the metric and the connection, which, for metric connections, are equivalent to the metric and the torsion. The action functional I^m for the matter fields is built up from both the matter fields, their spacetime-covariant derivatives and the 'spacetime' fields; while the action I^g for the latter, the 'gravitational action', is a functional only of the spacetime fields and their *coordinate*-derivatives. Two important consequences of this distinction are the following:

(1) The Lagrangian scenario for the matter fields yields, through the Noether identity, tensorial (but not necessarily internal-gauge independent) canonical Noether

currents $C^\mu[K]$ for any vector field K on M. These currents are built up from the *canonical* energy–momentum and spin tensors; and the very notion of energy–momentum and angular momentum is defined by these currents [1–3]. However for the spacetime fields a similar analysis yields only pseudotensorial or rigid-basis-dependent (i.e. spacetime-gauge dependent) quantities [4–8].

(2) One can take the variational derivative of I^m with respect to both the matter and spacetime fields, the former yields the field equations while the latter defines the *dynamical* energy–momentum and spin tensors. The Belinfante–Rosenfeld-like combinations of the *canonical* tensors give these dynamical ones [1–3]. The only possible variational derivative of I^g yields the field equations but we do not have any gravitational counterpart of the *dynamical* energy–momentum and spin tensors, even in principle. The dynamical energy–momentum and spin tensors of the matter fields become the 'source density' of the dynamical metric and connection. Thus if we want to give a complete as possible description of the energy–momentum and angular momentum properties of gravity the gravitational counterparts of *both* the *canonical* energy–momentum and spin tensors should be considered. (In Einstein's theory the Belinfante–Rosenfeld combination of the highly coordinate-dependent canonical energy–momentum and spin pseudotensors is minus the Einstein tensor [8].)

In Einstein's theory the gravitational canonical (pseudo) currents have two nice properties: (i) $C^\mu[K] + T^{\mu\nu}K_\nu$ is always (pseudo) conserved and can be derived from a (pseudo) tensorial superpotential, which can be expected since any vector field is a symmetry of the total matter + gravity action; and (ii) if K is a Killing vector of the geometry then $C^\mu[K]$ and $T^{\mu\nu}K_\nu$ are separately conserved, in accordance with our physical picture that a symmetry of the 'gravitational interaction' should imply the separate conservation of the corresponding matter and gravitational quantities.

The usage of these spacetime-gauge dependent canonical Noether (pseudo) currents, however, contradicts the principle of general covariance. To rule out this conflict several tensorial and gauge independent superpotentials have been proposed [9, 10]. However, conceptually the very notion of energy–momentum and angular momentum is connected to the Noether identity and hence the interpretation of the new tensorial conserved quantities is not obvious. Furthermore, neither are they quite satisfactory even from a pragmatic point of view, as they do not always yield the expected global energy–momentum and angular momentum of an asymptotically flat spacetime [5, 10].

1.1. The aim of the paper

A possible way out of this difficulty is to retain the notion of energy–momentum and angular momentum and to resolve the contradiction to the principle of general covariance. Since mathematically the theory of gravity is a theory of metric connections on the bundle of linear frames $L(M)$ over M [11], it seems natural first to try to reformulate the spacetime-gauge dependent quantities and formulae in terms of differential forms on $L(M)$. This reformulation may yield a unification of the different pseudotensorial and rigid-basis-dependent approaches into a single manifest gauge invariant formalism ('general relativity on the bundle of linear frames'). (If in the principle of general covariance the geometric objects by means of which the laws of nature should be able to be reformulated were not required to be geometric objects *on the spacetime manifold* M but only *on the manifold of frames of the spacetime*, i.e. on $L(M)$, then the contradiction to the principle of general covariance would have been resolved. Since the reformulability of a pseudotensorial quantity on M as a differential form on $L(M)$ is non-trivial, this revaluated form of the principle of general covariance would not be vacuous.) The energy–momentum and angular momentum density-like quantities *on* M remain gauge dependent as they are pull backs of non-horizontal forms along local sections of $L(M)$. If, however, we have a 2-co-dimensional closed submanifold $\mathcal{S}$ in M then, in contrast to internal gauge theories, $\mathcal{S}$ may be used to reduce the gravitational gauge freedom at the points of $\mathcal{S}$. Thus if we pull back a non-horizontal 'superpotential' form along preferred local sections of $L(M)$ we may obtain well defined quasi local energy–momentum and angular momentum expressions. The usage of these quasi local expressions would not contradict the principle of general covariance.

Our present paper is devoted to the differential geometric reformulation and unification of the previous different pseudotensorial and rigid-basis-dependent approaches. First, for the sake of completeness and to fix the notation, in section 1.2. the main differential geometric notions and formulae are reviewed, where, as far as is possible, the notation of Kobayashi and Nomizu [11] is used. We do not specify the dimension and the signature since certain properties and the structure of a specific theory can be understood more easily from a more general one (e.g. in four dimensions the dual of a 2-form is also a 2-form, thus if an integral should somehow be formed for a two-dimensional submanifold it is not *a priori* clear whether the 2-form or its dual should be used), moreover the formalism can be applied for lower and higher dimensional models, including the Euclidean ones. In subsections 2.1–2.3 Sparling's form [12] and its relation to various pseudotensors are discussed. In section 3 the relation between the Noether form and the gravitational Noether (pseudo) currents is discussed and then the Noether form is identified as the gravitational canonical Noether current within the framework of a Lagrangian scenario on the bundle $L(M)$. Finally, in section 4, the differential geometric form of the Belinfante–Rosenfeld equations and its implications for the Landau–Lifshitz pseudotensors are considered.

1.2. Conventions, notation and the mathematical background

Let M be an m-dimensional manifold, g a metric on M of signature $p-q$, $p+q=m$, let ∇_μ be the unique torsion-free covariant derivation determined by g and ε the natural volume m-form associated with g; i.e. if $(x^1,\dots,x^m)$ is a local coordinate system then $\varepsilon=\sqrt{|g|}\epsilon_{\alpha_1\dots\alpha_m}\mathrm{d}x^{\alpha_1}\wedge\dots\wedge\mathrm{d}x^{\alpha_m}=m!\sqrt{|g|}\mathrm{d}x^1\wedge\dots\wedge\mathrm{d}x^m$. ($\epsilon_{\alpha_1\dots\alpha_m}$ is the totally skew Levi–Civita symbol, $\epsilon_{1\dots m}=1$, Greek indexes are coordinate indexes and for the exterior product the convention compatible with $2\mathrm{d}x^1\wedge\mathrm{d}x^2=\mathrm{d}x^1\otimes\mathrm{d}x^2-\mathrm{d}x^2\otimes\mathrm{d}x^1$ is used. Abstract indexes will not be used in this paper.)

Let $L(M)$ be the linear frame bundle over M, $\{\delta_i\}$, $i=1,\dots,m$, be the standard basis for $\boldsymbol{R}^m$, i.e. $\delta_i=(0,\dots,0,1_i,0,\dots,0)$ and $\theta=\theta^i\delta_i$ the canonical $\boldsymbol{R}^m$-valued 1-form on $L(M)$. The metric g and the volume form ε of M define a set of functions on $L(M)$. If, for example, $w=(p,\{E_a\})\in L(M)$; i.e. w is a basis $\{E_a\}$ at T_pM, then $g_{ab}(w):=g_p(E_a,E_b)$ and $\varepsilon_{a_1\dots a_m}(w):=\varepsilon_p(E_{a_1},\dots,E_{a_m})$. One can define g^{ab} and $\varepsilon^{a_1\dots a_m}$ too, for which $\varepsilon_{a_1\dots a_r e_{r+1}\dots e_m}\varepsilon^{b_1\dots b_r e_{r+1}\dots e_m}=(-)^q(m-r)!\delta^{b_1\dots b_r}_{a_1\dots a_r}$.

For any $r=0,1,\dots,m$ let [12, 13]

$$\Sigma_{a_1\dots a_r}:=\frac{1}{(m-r)!}\varepsilon_{a_1\dots a_r e_{r+1}\dots e_m}\theta^{e_{r+1}}\wedge\dots\wedge\theta^{e_m}. \tag{1.2.1}$$

It is a tensorial $(m-r)$ form on $L(M)$, transforming according to the rth exterior power of the contragredient representation of $GL(m,R)$. If $r=0$ then this is just $\Sigma := (1/m!)\varepsilon_{e_1\dots e_m}\theta^{e_1}\wedge\ldots\wedge\theta^{e_m}$, while for $r=m$ this is the function $\varepsilon_{a_1\dots a_m}$. One can easily verify that

$$\theta^b\wedge\Sigma_{a_1\dots a_r} = (-)^{r+1} r\delta^b_{[a_1}\Sigma_{a_2\dots a_r]}. \tag{1.2.2}$$

Let $\omega^a{}_b$ be a connection 1-form on $L(M)$ compatible with the metric g. The structure equations for the torsion Ξ^a and curvature 2-form $\Omega^a{}_b$ are

$$\Xi^a = \mathrm{d}\theta^a + \omega^a{}_b\wedge\theta^b \tag{1.2.3}$$

$$\Omega^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_e\wedge\omega^e{}_b. \tag{1.2.4}$$

If index lowering and raising are defined by g_{ab} and g^{ab}, e.g. $\omega_{ab} := g_{ae}\omega^e{}_b$, then the condition of metric compatibility is

$$\mathrm{d}g_{ab} = \omega_{ab} + \omega_{ba}. \tag{1.2.5}$$

This implies $\mathrm{d}\varepsilon_{a_1\dots a_m} = \varepsilon_{a_1\dots a_m}\omega^e{}_e$ and, in general,

$$\mathrm{d}\Sigma_{a_1\dots a_r} = \Xi^a\wedge\Sigma_{a_1\dots a_r a} + (-)^{r+1} r\omega^a{}_{[a_1}\wedge\Sigma_{a_2\dots a_r]a}. \tag{1.2.6}$$

Because of the metric compatibility, $\Omega_{ab} := g_{ae}\Omega^e{}_b = -\Omega_{ba}$.

The first and the second Bianchi identities are

$$\mathrm{d}\Xi^a = \Omega^a{}_e\wedge\theta^e - \omega^a{}_e\wedge\Xi^e \tag{1.2.7}$$

$$\mathrm{d}\Omega^a{}_b = \Omega^a{}_e\wedge\omega^e{}_b - \omega^a{}_e\wedge\Omega^e{}_b. \tag{1.2.8}$$

In this formalism Einstein's tensor, $G^i{}_j := R^i{}_j - \frac{1}{2}R\delta^i_j$, is given by

$$-\tfrac{1}{2}\Omega^{ab}\wedge\Sigma_{iab} = G^j{}_i\Sigma_j. \tag{1.2.9}$$

The curvature tensor can be expressed by horizontal m-forms:

$$\Omega^a{}_b\wedge\Sigma_{cd} = R^a{}_{bcd}\Sigma \tag{1.2.10}$$

and hence so can both the Ricci tensor and the curvature scalar

$$\Omega^{aj}\wedge\Sigma_{ai} = R^j{}_i\Sigma \tag{1.2.11}$$

and

$$R\Sigma = \Omega^{ab}\wedge\Sigma_{ab} = \mathrm{d}\left(\omega^{ab}\wedge\Sigma_{ab}\right) - \omega^a{}_e\wedge\omega^{eb}\wedge\Sigma_{ab} + \omega^{ab}\wedge\Xi^e\wedge\Sigma_{abe} \tag{1.2.12}$$

respectively. Since the horizontal subspaces are m-dimensional and Ω^{ab}, Ξ^c and Σ_{abc} are all horizontal,

$$\mathrm{d}\left(\Omega^{ab}\wedge\Sigma_{ab}\right) = \Omega^{ab}\wedge\Xi^c\wedge\Sigma_{abc} = 0; \tag{1.2.13}$$

i.e. $\Omega^{ab}\wedge\Sigma_{ab}$ is closed.

On $L(M)$ $\{\omega^a{}_b,\theta^e\}$ and $\{D_m{}^n, B(\delta_k)\}$ form a dual basis, where $D_m{}^n$ is the fundamental vector field associated with the element $e_m{}^n$ of the Weyl basis of $gl(m,\boldsymbol{R})$ and $B(\delta_k)$ is the kth standard horizontal vector field. Thus any vector field X on $L(M)$ has a unique decomposition $X=(hX)^eB(\delta_e)+(vX)^a{}_bD_a{}^b$. Using (1.2.5) and (1.2.8), the Lie derivative of Ω^{ab} along X takes the form

$$Ł_X\Omega^{ab}=\mathrm{d}(\iota_X\Omega^{ab})-\iota_X\Omega^{ae}\wedge\omega^b{}_e+\iota_X\Omega^{be}\wedge\omega^a{}_e-(vX)^a{}_e\Omega^{eb}+(vX)^b{}_e\Omega^{ea} \tag{1.2.14}$$

and, using (1.2.6), the Lie derivative of Σ_{ab} is

$$Ł_X\Sigma_{ab}=\Big(\mathrm{d}(hX)^e+\omega^e{}_f(hX)^f\Big)\wedge\Sigma_{eab}+\Big((vX)^e{}_b\Sigma_{ae}-(vX)^e{}_a\Sigma_{be}\Big) \\ +\iota_X(\Xi^e\wedge\Sigma_{eab})+(hX)^e\Xi^c\wedge\Sigma_{eabc}. \tag{1.2.15}$$

Here $\mathrm{D}(hX)^e:=\mathrm{d}(hX)^e+\omega^e{}_f(hX)^f$ is the so-called covariant exterior derivative of $(hX)^e$. Finally, using (1.2.13), one has

$$Ł_X(\Omega^{ab}\wedge\Sigma_{ab})=\mathrm{D}(hX)^e\wedge\Omega^{ab}\wedge\Sigma_{eab}+\mathrm{d}(\iota_X\Omega^{ab}\wedge\Sigma_{ab})+(hX)^e\Omega^{ab}\wedge\Xi^c\wedge\Sigma_{eabc}. \tag{1.2.16}$$

For torsion-free connections the last term vanishes and, by virtue of (1.2.13), the first term on the right becomes an exact form.

A local section $s:U\to L(M)$ is a field of basis vectors $\{\boldsymbol{E}_a\}$ on U. (More precisely, the vector $\boldsymbol{E}_a$ at $p\in U$ is the element $s(p)(\delta_a)$ of the vector bundle $T(M)$, associated with $s(p)\in L(M)$ and $\delta_a\in\boldsymbol{R}^m$.) The pull back $s^*(\theta^a)$ is a field of basis 1-forms on U, dual to the vector basis. The structure coefficients of the section is the collection $\{c^a_{rs}\}$ of functions defined on U by $[\boldsymbol{E}_r,\boldsymbol{E}_s]=c^a_{rs}\boldsymbol{E}_a$. s is called the coordinate or holonomic section if there is a coordinate system $(x^1,\dots,x^m)$ on U such that $\boldsymbol{E}_a=\partial/\partial x^a$, $a=1,\dots,m$, and then $s^*(\theta^a)=\mathrm{d}x^a$. s is a coordinate section iff $c^a_{rs}=0$. s is called rigid or anholonomic with respect to the metric g if for the pull backs $\vartheta^a:=s^*(\theta^a)$ and for some constant matrix η_{ab} we have $g=\vartheta^a\otimes\vartheta^b\eta_{ab}$. The pull back $s^*(\omega^a{}_b)$ is a $gl(m,\boldsymbol{R})$-valued 1-form on U, thus it can be expressed in the naturally defined basis of 1-forms $\{s^*(\theta^a)\}$: $s^*(\omega^a{}_b)=:\omega^a_{rb}s^*(\theta^r)$. If s is a coordinate section then for torsion-free connections Γ^a_{rb}, defined by $s^*(\omega^a{}_b)=:\Gamma^a_{rb}\mathrm{d}x^r$, are the usual Christoffel symbols; while if s is a rigid section then γ^a_{rb}, defined by $s^*(\omega^a{}_b)=:\gamma^a_{rb}\vartheta^r$, are the Ricci rotation coefficients and $\gamma^a_{rb}=-\gamma^e_{rf}\eta_{eb}\eta^{fa}$. For torsion-free connections $\gamma^a_{rb}=\frac{1}{2}(c^a_{rb}+\eta^{ae}c^f_{er}\eta_{fb}+\eta^{ae}c^f_{eb}\eta_{fr})$ and $c^a_{rb}=\gamma^a_{rb}-\gamma^a_{br}$. Twice the pull back of the curvature form is just the curvature tensor: $2s^*(\Omega^a{}_b)=R^a{}_{brs}s^*(\theta^r)\wedge s^*(\theta^s)$, and the pull back $s^*(\Sigma)$ is $1/m!$ times the volume form ε on M.

For torsion-free connections and $\kappa>0$ let us define the following m-forms on $L(M)$:

$$\Lambda_{\mathrm{H}}:=\frac{1}{2\kappa}\Omega^{ab}\wedge\Sigma_{ab} \tag{1.2.17}$$

$$\Lambda_{\mathrm{E}}:=\frac{1}{2\kappa}\Big(\Omega^{ab}\wedge\Sigma_{ab}-\mathrm{d}(\omega^{ab}\wedge\Sigma_{ab})\Big)=-\frac{1}{2\kappa}\omega^a{}_e\wedge\omega^{eb}\wedge\Sigma_{ab}. \tag{1.2.18}$$

Because of (1.2.13) both $\Lambda_{\rm H}$ and $\Lambda_{\rm E}$ are closed. If $s : U \to L(M)$ is any local section then $s^*(\Lambda_{\rm H})$ is $(1/m!)\varepsilon$ times the Hilbert Lagrangian. It is given in the holonomic/anholonomic description if s is a holonomic/anholonomic section. Since the pull backs of $\mathrm{d}(\omega^{ab}\wedge\Sigma_{ab})$ give the total divergences left from Hilbert's Lagrangian to obtain the familiar first-order ones, the pull back of $\Lambda_{\rm E}$ along a holonomic section gives Einstein's Lagrangian [4, 5] $\mathcal{L}_{\rm hE}$: $s^*(\Lambda_{\rm E}) = (1/\sqrt{|g|})\mathcal{L}_{\rm hE}s^*(\Sigma)$; while for anholonomic s it is the first order Møller–Nester Lagrangian [6, 7] $\mathcal{L}_{\rm aE}$: $s^*(\Lambda_{\rm E}) = (1/\sqrt{|g|}\mathcal{L}_{\rm aE}s^*(\Sigma)$.

2. Sparling's form and the energy–momentum pseudotensors

Let us define the Nester–Witten form [14, 15] as

$$u_i := -\tfrac{1}{2}\omega^{ab}\wedge\Sigma_{iab}. \tag{2.1}$$

This form is an $\boldsymbol{R}^{m*}$-valued pseudotensorial $(m-2)$-form on $L(M)$ which transforms according to the contragredient representation of $GL(m,\boldsymbol{R})$. (Thus for the dimension of M we necessarily have $m \geqslant 3$.) The terms in its exterior derivative can naturally be grouped as

$$\mathrm{d}u_i = -\tfrac{1}{2}\Omega^{ab}\wedge\Sigma_{iab} + \tfrac{1}{2}\Xi^c\wedge\omega^{ab}\wedge\Sigma_{iabc} + t_i \tag{2.2}$$

where

$$t_i := -\tfrac{1}{2}\left(\omega^e{}_i\wedge\omega^{ab}\wedge\Sigma_{eab} + \omega^a{}_e\wedge\omega^{eb}\wedge\Sigma_{iab}\right) \tag{2.3}$$

is Sparling's $(m-1)$-form [12, 13]. What is interesting here is the structure of the right-hand side of equation (2.2), the Sparling equation: the curvature appears through the Einstein tensor, which is the only horizontal term on the right, the second term is linear and the third is quadratic in the connection form. Thus t_i is only pseudotensorial, transforming according to the contragredient representation of $GL(m,\boldsymbol{R})$. It might be interesting to note that the covariant exterior derivative $\mathrm{D}u_i$ of u_i is just the first term on the right of (2.2), thus for torsion-free connections the Sparling form is the 'correction' to $\mathrm{d}u_i$ to become the tensorial $\mathrm{D}u_i$. The importance of u_i and t_i in general relativity is shown by the following theorem, due to Sparling [12] for the vacuum case and Dubois-Violette and Madore [13] for the general case:

Theorem 2.1. For any $\boldsymbol{R}^{m*}$-valued horizontal $(m-1)$ form T_i satisfying $\mathrm{D}T_i := \mathrm{d}T_i - \omega^e{}_i\wedge T_e = 0$ and $\kappa\in\boldsymbol{R}$ the following statements are equivalent:
(1) $\omega^a{}_b$ is torsion free, $\Xi^a = 0$, and $\frac{1}{2}\Omega^{ab}\wedge\Sigma_{iab} + \kappa T_i = 0$;
(2) $\kappa T_i + t_i = \mathrm{d}u_i$; and
(3) $\mathrm{d}(\kappa T_i + t_i) = 0$.

This theorem gives an alternative formulation of Einstein's theory: a metric connection on $L(M)$ is torsion free and satisfies Einstein's equations with matter energy–momentum tensor $T^j{}_i$, defined by $T_i =: T^j{}_i\Sigma_j$, iff the Sparling and the Nester–Witten forms satisfy condition (2); which is equivalent to the Sparling form satisfying condition (3). In Einstein's theory (3) looks like as a conservation equation, while (2) gives us the 'superpotential' for the conserved quantity $\kappa T_i + t_i$: it is just the Nester–Witten form. But since these quantities are defined in $L(M)$ instead of M, and, moreover t_i and u_i are only pseudotensorial forms, exterior equations (2) and (3) yield equations in M only if we pull them back along a local section of $L(M)$. In fact, Frauendiener [16] calculated the pull back of u_i and $\mathrm{d}u_i$ along a coordinate section of $L(M)$ by means of which he could show that $s^*(t_i)$ is essentially Einstein's canonical energy–momentum pseudotensor; moreover the pull back of the contravariant form of Sparling's form was identified as the Landau–Lifshitz pseudotensor. However, we would also like to recover energy–momentum pseudotensors in the rigid basis or anholonomic formulation of general relativity. Thus we calculate the pull backs first along a general section and then specialize s to be a coordinate and then a rigid section. This will be done in the following paragraph. One can also take the contravariant and dual forms of the Nester–Witten and Sparling forms, whose pull backs will be considered in sections 2.2 and 2.3. It turns out that the pull back of the contravariant form of Sparling's form yields Bergmann's pseudotensor, while the pull back of the dual form of Sparling's form gives the Landau–Lifshitz pseudotensor.

2.1. *The canonical energy–momentum pseudotensors*

The pull back of the Nester–Witten form along a general section $s : U \to L(M)$ is

$$s^*(u_i) = -\tfrac{1}{2}\left(\omega^{[a}_{ie} g^{b]e} + \delta^{[a}_i \omega^{b]}_{rs} g^{rs} - \delta^{[a}_i g^{b]e} \omega^r_{re}\right) s^*(\Sigma_{ab}). \qquad (2.1.1)$$

If $(x^\mu) = (x^1, \ldots, x^m)$ is a local coordinate system on U and s is the corresponding coordinate section then $E^\mu_a = \delta^\mu_a$ (and hence there is no difference between the Greek and Latin indexes), $\omega^a_{rb} = \Gamma^a_{(rb)}$ and

$$s^*(u_i) = \frac{1}{4\sqrt{|g|}} \mathsf{U}_i{}^{ab} s^*(\Sigma_{ab}). \qquad (2.1.2)$$

Here $\mathsf{U}_i{}^{ab}$ is the well known von Freud superpotential, which can also be given as $\mathsf{U}_\beta{}^{\alpha\mu} = (1/\sqrt{|g|} g_{\beta\rho} \partial_\nu (|g| G^{\alpha\mu\rho\nu})$, $G^{\alpha\mu\beta\nu} := g^{\alpha\beta} g^{\mu\nu} - g^{\alpha\nu} g^{\beta\mu}$ [4, 5]. If s is a rigid section on U then $\omega^a_{rb} = \gamma^a_{rb}$ and

$$s^*(u_i) = \tfrac{1}{4} E^\mu_i \mathsf{V}_\mu{}^{\alpha\beta} s^*(\Sigma_{\alpha\beta}) \qquad (2.1.3)$$

where $\mathsf{V}_\mu{}^{\alpha\beta}$ is the *tensorial* superpotential of Møller [6] and Goldberg [5]. Thus the components of the pull backs of u_i are the dual of certain superpotentials of general relativity.

Now consider the pull backs of condition (2) of theorem 2.1, i.e. Sparling's equation in general relativity. Since T_i is horizontal it has the form $T^j{}_i \Sigma_j$, and hence $s^*(T_i) = T^j{}_i s^*(\Sigma_j)$ is independent of the section s. If for brevity $u_i{}^{ab}$ is defined by $s^*(u_i) =: u_i{}^{ab} s^*(\Sigma_{ab})$ then

$$\begin{aligned} s^*(\mathrm{d}u_i) &= \mathrm{d}u_i{}^{ab} \wedge s^*(\Sigma_{ab}) + u_i{}^{ab} s^*(\mathrm{d}\Sigma_{ab}) \\ &= -2\left(E^\mu_a \partial_\mu u_i{}^{ae} + u_i{}^{ab} \omega^e_{ab} + u_i{}^{ae} \omega^r_{ra}\right) s^*(\Sigma_e). \end{aligned} \qquad (2.1.4)$$

The pull back of Sparling's form along a general section s is

$$s^*(t_i) = -\tfrac{1}{2}\Big(\delta^j_i(\omega^e_{ek}\omega^k_{rs} - \omega^e_{rk}\omega^k_{es})g^{rs} + g^{jk}\omega^s_{sk}\omega^r_{ri} - g^{jk}\omega^l_{ik}\omega^e_{el} - \omega^e_{ei}\omega^j_{rs}g^{rs} \\ - \omega^j_{ri}\omega^e_{es}g^{rs} + \omega^j_{rk}\omega^k_{is}g^{rs} + \omega^j_{kr}\omega^k_{si}g^{rs} \\ + (\omega^j_{ki} - \omega^j_{ik})\omega^k_{rs}g^{rs} + (\omega^k_{il} - \omega^k_{li})\omega^l_{kr}g^{rj}\Big)s^*(\Sigma_j). \tag{2.1.5}$$

If s is a coordinate section then

$$s^*(t_i) = \kappa_{\mathrm{hE}}t^j{}_i s^*(\Sigma_j) \tag{2.1.6}$$

where

$$_{\mathrm{hE}}t^\alpha{}_\beta := \frac{1}{\sqrt{|g|}}\left(\mathcal{L}_{\mathrm{hE}}\delta^\alpha_\beta - \frac{\partial\mathcal{L}_{\mathrm{hE}}}{\partial\partial_\alpha g_{\mu\nu}}\partial_\beta g_{\mu\nu}\right)$$

is Einstein's canonical energy–momentum pseudotensor [4, 5], and $s^*(\mathrm{d}u_i) = (1/2\sqrt{|g|})\partial_a \cup_i{}^{ja}\ s^*(\Sigma_j)$. Thus the pull back of Sparling's equation along a coordinate section is equivalent to the von Freud equation $\kappa\sqrt{|g|}\,(T^\alpha{}_\beta + {}_{\mathrm{hE}}t^\alpha{}_\beta) = \frac{1}{2}\partial_\mu \cup_\beta{}^{\alpha\mu}$ [16]. If s is a rigid section then

$$\begin{aligned} s^*(t_i) &= \left(\kappa_{\mathrm{aE}}t^\mu{}_\nu E^\nu_i + \tfrac{1}{2}\vee_\rho{}^{\mu\nu}\partial_\nu E^\rho_i\right)s^*(\Sigma_\mu) \\ &= \left(\kappa_{\mathrm{AE}}\theta^\mu{}_\nu E^\nu_i + \tfrac{1}{2}\vee_\rho{}^{\mu\nu}\nabla_\nu E^\rho_i\right)s^*(\Sigma_\mu) \end{aligned} \tag{2.1.7}$$

where

$$_{\mathrm{aE}}t^\alpha{}_\beta := \frac{1}{\sqrt{|g|}}\left(\mathcal{L}_{\mathrm{aE}}\delta^\alpha_\beta - \frac{\partial\mathcal{L}_{\mathrm{aE}}}{\partial\partial_\alpha\vartheta^r_\rho}\partial_\beta\vartheta^r_\rho\right)$$

is the canonical energy–momentum pseudotensor in the anholonomic description [5, 6, 8] and $_{\mathrm{AE}}\theta^\alpha{}_\beta$ is the gravitational energy–momentum *tensor* defined in [8] by

$$_{\mathrm{AE}}\theta^\alpha{}_\beta := {}_{\mathrm{aE}}t^\alpha{}_\beta - \frac{1}{2\kappa}\Gamma^\rho_{\mu\beta}\vee_\rho{}^{\alpha\mu} \tag{2.1.8}$$

and satisfying

$$G^\alpha{}_\beta + \kappa_{\mathrm{AE}}\theta^\alpha{}_\beta = \tfrac{1}{2}\nabla_\mu\vee_\beta{}^{\alpha\mu}. \tag{2.1.9}$$

Thus although $s^*(t_i)$ for a rigid section is *tensorial*, it deviates from the naturally defined gravitational energy–momentum (pseudo) tensors. Therefore the various energy–momentum expressions are not simply pull backs of a single geometric object, e.g. t_i, along various sections. However, $s^*(\mathrm{d}u_i) = \frac{1}{2}\nabla_\rho\left(E^\mu_i\vee_\mu{}^{\nu\rho}\right)s^*(\Sigma_\nu)$, thus the pull back of Sparling's equation along a rigid section is equivalent to the superpotential equation (2.1.9).

Finally, the pull backs of the 'conservation equation' (3) of theorem 2.1 are just the (pseudo) divergence equations: $\partial_\alpha\left(\sqrt{|g|}(G^\alpha{}_\beta + \kappa_{\mathrm{hE}}t^\alpha{}_\beta)\right) = 0$ and $\nabla_\alpha\left(G^\alpha{}_\beta + \kappa_{\mathrm{AE}}\theta^\alpha{}_\beta\right) = 0$, respectively.

2.2. *The contravariant form of the canonical pseudotensors*

The contravariant form of the Nester–Witten form will be defined by $u^i := g^{ij}u_j$; and clearly its pull backs along coordinate and rigid sections give the contravariant form of the von Freud and the Møller–Goldberg superpotentials: $g^{\beta\rho}\,\mathsf{U}_\rho{}^{\alpha\mu}$ and $g^{\beta\rho}\,\mathsf{V}_\rho{}^{\alpha\mu}$, respectively.

In Einstein's theory the exterior derivative of u^i is the sum of the horizontal $(m-1)$ form $\kappa T^i := \kappa g^{ij}T_j$ and terms quadratic in the connection forms:

$$\mathrm{d}u^i = \mathrm{d}g^{ij}\wedge u_j + g^{ij}\mathrm{d}u_i = \kappa T^i + \Theta^i \tag{2.2.1}$$

where

$$\Theta^i := t^i - \left(\omega^{ij} + \omega^{ji}\right)\wedge u_j. \tag{2.2.2}$$

The pull back of $\mathrm{d}u^i$ along a general section is

$$s^*(\mathrm{d}u^i) = s^*(\mathrm{d}u_j)g^{ji} - 2\left(\omega^i_{rs}g^{sj} + \omega^j_{rs}g^{si}\right)u_j{}^{re}s^*(\Sigma_e) \tag{2.2.3}$$

while if for brevity $\tau^j{}_i$ is defined by $s^*(t_i) =: \tau^j{}_i s^*(\Sigma_j)$ then

$$s^*(\Theta^i) = \left(\tau^e{}_k g^{ki} - 2(\omega^i_{rs}g^{sj} + \omega^j_{rs}g^{si})u_j{}^{re}\right)s^*(\Sigma_e). \tag{2.2.4}$$

If s is a coordinate section then

$$s^*\left(\Theta^i\right) = \kappa_{\mathrm{hE}}\theta^{ji}s^*(\Sigma_j) \tag{2.2.5}$$

where

$${}_{\mathrm{hE}}\theta^{\alpha\beta} := {}_{\mathrm{hE}}t^{\alpha\beta} + \frac{1}{2\kappa\sqrt{|g|}}\partial_\mu g^{\beta\rho}\,\mathsf{U}_\rho{}^{\alpha\mu}$$

is the contravariant form of Einstein's energy–momentum (i.e. Bergmann's) pseudotensor; and the pull back of (2.2.1) gives the superpotential equation for ${}_{\mathrm{hE}}\theta^{\alpha\beta}$. If s is a rigid section then

$$s^*(\Theta^i) = s^*(t_e)\eta^{ei} = \left(\kappa_{\mathrm{AE}}\theta^{\alpha\beta}\vartheta^i_\beta + \tfrac{1}{2}\mathsf{V}_\rho{}^{\alpha\mu}\nabla_\mu E^\rho_k\eta^{ki}\right)s^*(\Sigma_\alpha). \tag{2.2.6}$$

Thus $s^*(\Theta^i)$, which is *tensorial* again, deviates from $\kappa_{\mathrm{AE}}\theta^{\alpha\beta}\vartheta^i_\beta s^*(\Sigma_\alpha)$, while the pull back of (2.2.1) gives the contravariant form of (2.1.9). (One can introduce the contravariant form ${}_{\mathrm{aE}}\theta^{\alpha\beta}$ of ${}_{\mathrm{aE}}t^\alpha{}_\beta$ too, which is only *pseudotensorial* and not to be confused with the *tensorial* ${}_{\mathrm{AE}}\theta^{\alpha\beta}$ defined in the previous paragraph.)

2.3. The dual form of the canonical pseudotensors

Let us define the dual form of the Nester–Witten form by

$$u_{e_2\ldots e_m} := u^e \varepsilon_{ee_2\ldots e_m} \tag{2.3.1}$$

and let $T_{e_2\ldots e_m} := T^e \varepsilon_{ee_2\ldots e_m}$. Then

$$\mathrm{d}u_{e_2\ldots e_m} = \kappa T_{e_2\ldots e_m} + \Theta_{e_2\ldots e_m} \tag{2.3.2}$$

where

$$\Theta_{e_2\ldots e_m} := \left(t^e + (g^{ef}\omega^k{}_k - \omega^{ef} - \omega^{fe}) \wedge u_f\right) \varepsilon_{ee_2\ldots e_m}. \tag{2.3.3}$$

Now we are interested only in *coordinate sections*, when the pull back of $u_{e_2\ldots e_m}$ is

$$\begin{aligned} s^*(u_{e_2\ldots e_m}) &= \frac{1}{4\sqrt{|g|}} \cup_f{}^{ab} g^{fe} \varepsilon_{ee_2\ldots e_m} s^*(\Sigma_{ab}) \\ &= \frac{1}{4}\partial_r \left(|g| G^{aber}\right) \epsilon_{ee_2\ldots e_m} \frac{1}{(m-2)!} \epsilon_{abf_3\ldots f_m} \mathrm{d}x^{f_3} \wedge \ldots \wedge \mathrm{d}x^{f_m} \end{aligned} \tag{2.3.4}$$

and hence

$$\begin{aligned} s^*(\mathrm{d}u_{e_2\ldots e_m}) &= \frac{1}{2}\partial_r\partial_s \left(|g| G^{fres}\right) \frac{1}{(m-1)!} \epsilon_{ee_2\ldots e_m} \epsilon_{ff_2\ldots f_m} \mathrm{d}x^{f_2} \wedge \ldots \wedge \mathrm{d}x^{f_m} \\ &= \frac{1}{2|g|}\partial_r\partial_s \left(|g| G^{fres}\right) \frac{1}{(m-1)!} \varepsilon_{ee_2\ldots e_m} \varepsilon_{ff_2\ldots f_m} \mathrm{d}x^{f_2} \wedge \ldots \wedge dx^{f_m}. \end{aligned} \tag{2.3.5}$$

Since the pull back of $\mathrm{d}u_{e_2\ldots e_m}$ is the double dual of the symmetric object defining the Landau–Lifshitz pseudotensor [17],

$$\kappa_{\mathrm{LL}} t^{\alpha\beta} + G^{\alpha\beta} := \frac{1}{2|g|}\partial_\mu\partial_\nu(|g| G^{\alpha\mu\beta\nu})$$

one may expect that the pull back of $\Theta_{e_2\ldots e_m}$ is just the double dual of the Landau–Lifshitz pseudotensor. In fact, the pull back of (2.3.3) along a coordinate section is

$$s^*(\Theta_{e_2\ldots e_m}) = \kappa_{\mathrm{LL}} t^{fe} \varepsilon_{ee_2\ldots e_m} s^*(\Sigma_f). \tag{2.3.6}$$

One can take various forms of the Nester–Witten form and hence the Sparling equation, and one can then pull them back along various local sections of $L(M)$, yielding different superpotentials and pseudotensors. However, the mathematical content of all these quantities and equations is the same: the sum of the 'Einstein $(m-1)$ form' $-\frac{1}{2}\Omega^{ab} \wedge \Sigma_{iab}$ and the Sparling form t_i is exact, it is the exterior derivative of the Nester–Witten form. From a physical point of view, however, these may differ in significance: for example if K_β is any vector field satisfying $\partial_{(\alpha} K_{\beta)} = 0$ then for the Landau–Lifshitz pseudotensor we have

$$\partial_\alpha\Big(|g|(G^{\alpha\beta} + \kappa_{\mathrm{LL}} t^{\alpha\beta}) K_\beta\Big) = \tfrac{1}{2}\partial_\mu\partial_\nu(|g| G^{\alpha\mu\beta\nu})\partial_{(\alpha} K_{\beta)} = 0.$$

What is (globally) conserved here is therefore the integral

$$\frac{1}{(m-1)!}\int\sqrt{|g|}\left(G^{\alpha\beta}+\kappa_{\rm LL}t^{\alpha\beta}\right)K_\beta\sqrt{|g|}\epsilon_{\alpha\gamma_2\ldots\gamma_m}\,{\rm d}x^{\gamma_2}\wedge\ldots\wedge{\rm d}x^{\gamma_m}$$

for an $(m-1)$-dimensional submanifold. However this is *not* the energy–momentum of the matter + gravity system even if $\partial_\alpha K_\beta = 0$, since we have an extra $\sqrt{|g|}$ coefficient not only in front of the gravitational part, but in front of the matter part also. If the extra $\sqrt{|g|}$ factor were in front of the gravitational term only but the matter term had the right coefficient then the matter part could be interpreted, for example, as energy–momentum or angular momentum and, in contrast to the strange feature of the gravitational part, would suggest the interpretation of the gravitational part also. The result would be surprising but acceptable [18]. The extra $\sqrt{|g|}$ in front of the matter part, however, plays the role of a coordinate-dependent weight function and destroys the clear interpretation of its integral. Thus it is hard to interpret these conserved quantities, in contrast e.g. to the integral of the Noether currents built up from ${}_{\rm AE}\theta^{\alpha\beta}$ and ${}_{\rm AE}\sigma^{\mu\alpha\beta}$ below [8]. Moreover if K is a Killing vector of the geometry then, in general, the pseudocurrent $|g|\left(G^{\alpha\beta}+\kappa_{\rm LL}t^{\alpha\beta}\right)K_\beta$ is *not* the sum of separately conserved (pseudo) currents, which could be expected on physical grounds, while the Noether currents just mentioned are. Perhaps the Landau–Lifshitz pseudocurrents above should be completed by spin parts, but, since the Landau–Lifshitz pseudotensor is not a canonical pseudotensor, it is not *a priori* clear how these spin parts should be defined. In the next two sections we return to this question and construct the missing spin part and discuss the Landau–Lifshitz pseudotensor further.

3. The canonical Noether current

Since only the energy–momentum pseudotensors of the holonomic description are seen to be recoverable from (various forms of) Sparling's form and the (pseudo) tensors of the anholonomic description systematically deviate from the pull backs of Sparling's form, one might be slightly frustrated and dissatisfied. Recall, however, that in the classical Lagrangian theory of matter fields, instead of the canonical energy–momentum tensor, it is the canonical Noether current, associated with a vector field on the spacetime, that has direct physical meaning. This Lagrangian scenario can also be applied for general relativity and one can construct the so-called canonical Noether (pseudo) currents as well [8].

Here we first show that there is a real valued $(m-1)$ form on $L(M)$, the Noether form, whose pull backs are the corresponding canonical Noether (pseudo) currents of gravity even if the local section is rigid. Then the dual form of the Noether form will be introduced, whose pull back tells us how to define the 'canonical Noether pseudocurrent' for the non-canonical Landau–Lifshitz pseudotensor. Finally, an even less pragmatic section follows, where we show that the Noether form is just the canonical Noether current in the sense of the scenario of the Lagrangian field theory on $L(M)$.

3.1. The Noether form on $L(M)$

Let K be any vector field on M and $\{K^a\}$ be the collection of functions on $L(M)$ defined by K: if $w=(p,\{E_a\})\in L(M)$ then let $K^a(w)$ be the ath component of

K in the basis $\{E_a\}$ at $T_p M$. (In the language of Kobayashi and Nomizu [11] $\{K^a\}$ is a zero form on $L(M)$ of type $(\boldsymbol{R}^m, GL(m,\boldsymbol{R}))$.) Thus $\text{Ł}_{D_m{}^n} K_a = \delta^n_a K_m$. K is a conformal Killing vector on M iff $\text{Ł}_{B(\delta_a)} K_b + \text{Ł}_{B(\delta_b)} K_a = \phi g_{ab}$ for some $GL(m,\boldsymbol{R})$-invariant function ϕ on $L(M)$; and K is a Killing vector iff $\phi = 0$.

The gravitational Noether form, associated with K, is defined by

$$\begin{aligned} C[K] := t_a K^a + \mathrm{d}K^a \wedge u_a = \Theta^a K_a + \mathrm{d}K_a \wedge u^a \\ = \left(\Theta^a + \omega^a{}_e \wedge u^e\right) K_a + \mathrm{D}K_a \wedge u^a. \end{aligned} \tag{3.1.1}$$

Then trivially

$$C[K] + \kappa T^a K_a = \mathrm{d}(K_a u^a) \tag{3.1.2}$$

implying $\mathrm{d}(C[K] + \kappa T^a K_a) = 0$. Because of

$$\begin{aligned} \mathrm{d}C[K] &= -\kappa(\mathrm{d}K_a \wedge T^a + K_a \mathrm{d}T^a) = -\kappa T^{ab} \mathrm{D}K_b \wedge \Sigma_a \\ &= -\kappa \text{Ł}_{B(\delta_a)} K_b T^{ab} \Sigma \end{aligned}$$

and the symmetry $T^{ab} = T^{(ab)}$, implied by the symmetry of Einstein's tensor in absence of torsion, $C[K]$ and $T^a K_a$ are separately closed for Killing vectors. For traceless matter energy–momentum tensor, $T^a{}_a = 0$, they are closed even for conformal Killing vectors too.

The pull back of $C[K]$ along a general local section of $L(M)$ is

$$\begin{aligned} s^*(C[K]) &= s^*(\Theta^a) K_a + s^*(\mathrm{d}K_a) \wedge s^*(u^a) \\ &= s^*(\Theta^a) K_a + E^\mu_a \partial_\mu K_b s^*(\theta^a \wedge u^b). \end{aligned} \tag{3.1.3}$$

If s is a coordinate section then

$$s^*(C[K]) = \kappa({}_{\mathrm{hE}}\theta^{\mu\nu} K_\nu + {}_{\mathrm{hE}}\sigma^{\mu\alpha\beta} \partial_\alpha K_\beta) s^*(\Sigma_\mu) \tag{3.1.4}$$

which is just the canonical Noether pseudocurrent in the *holonomic description:*

$$\begin{aligned} {}_{\mathrm{hE}}C^\mu[K] &:= {}_{\mathrm{hE}}\theta^{\mu\nu} K_\nu + \left({}_{\mathrm{hE}}\sigma^{\mu[\alpha\beta]} + {}_{\mathrm{hE}}\sigma^{\alpha[\beta\mu]} + {}_{\mathrm{hE}}\sigma^{\beta[\alpha\mu]}\right) \partial_\alpha K_\beta \\ &= {}_{\mathrm{hE}}\theta^{\mu\nu} K_\nu + {}_{\mathrm{hE}}\sigma^{\mu\alpha\beta} \partial_\alpha K_\beta \end{aligned}$$

where ${}_{\mathrm{hE}}\sigma^{\mu\alpha\beta} = -(1/2\kappa\sqrt{|g|}) g^{\beta\rho} \cup_\rho{}^{\alpha\mu}$ is the contravariant form of the canonical spin pseudotensor, satisfying the

$$\sqrt{|g|}_{\mathrm{hE}}\theta^{[\alpha\beta]} = -\partial_\mu \left(\sqrt{|g|}_{\mathrm{hE}}\sigma^{\mu[\alpha\beta]}\right)$$

algebraic Belinfante–Rosenfeld-type equation. If s is a rigid section then

$$s^*(C[K]) = \kappa\left({}_{\mathrm{AE}}\theta^{\mu\nu} K_\nu + {}_{\mathrm{AE}}\sigma^{\mu\alpha\beta} \nabla_\alpha K_\beta\right) s^*(\Sigma_\mu) \tag{3.1.5}$$

which gives the canonical Noether current of the anholonomic description:

$$
\begin{aligned}
{}_{\mathrm{AE}}C^{\mu}[K] &:= {}_{\mathrm{AE}}\theta^{\mu\nu}K_{\nu} + \left({}_{\mathrm{AE}}\sigma^{\mu[\alpha\beta]} + {}_{\mathrm{AE}}\sigma^{\alpha[\beta\mu]} + {}_{\mathrm{AE}}\sigma^{\beta[\alpha\mu]}\right)\nabla_{\alpha}K_{\beta} \\
&= {}_{\mathrm{AE}}\theta^{\mu\nu}K_{\nu} + {}_{\mathrm{AE}}\sigma^{\mu\alpha\beta}\nabla_{\alpha}K_{\beta}.
\end{aligned}
$$

Here ${}_{\mathrm{AE}}\sigma^{\mu\alpha\beta} = -(1/2\kappa)g^{\beta\rho}\vee_{\rho}{}^{\alpha\mu}$ is the contravariant form of the canonical spin *tensor* and satisfies the pair of *tensorial* Belinfante–Rosenfeld-type equations [8]:

$$
\begin{aligned}
&{}_{\mathrm{AE}}\theta^{[\alpha\beta]} = -\nabla_{\mu}\,{}_{\mathrm{AE}}\sigma^{\mu[\alpha\beta]} \\
&\nabla_{\nu}\,{}_{\mathrm{AE}}\theta^{\nu\mu} = -R^{\mu}{}_{\nu\alpha\beta}\,{}_{\mathrm{AE}}\sigma^{\nu\alpha\beta}.
\end{aligned}
$$

Thus although the pull backs of t_i and Θ^i along rigid sections are not exactly ${}_{\mathrm{AE}}\theta^{\alpha}{}_{\beta}\vartheta^{a}_{\alpha}E^{\beta}_{b}$ and ${}_{\mathrm{AE}}\theta^{\alpha\beta}\vartheta^{a}_{\alpha}\vartheta^{b}_{\beta}$, respectively, the pull backs of the Noether form are always the Noether (pseudo) currents. The Noether form therefore seems to be the geometric object on $L(M)$ which, with appropriately chosen vector fields $\boldsymbol{K}$, describes the momentum–angular momentum distribution of gravity.

Integrating a pull back of (3.1.2) for an $(m-1)$-dimensional submanifold with boundary one obtains the so-called 'global conservation equations'.

3.2. The dual form of the Noether form

Let us define the following $\wedge^m\boldsymbol{R}^m$-valued $(m-1)$ form on $L(M)$ for any vector field $\boldsymbol{K}$ on M:

$$
\begin{aligned}
C_{e_1\ldots e_m}[\boldsymbol{K}] &:= K_{[e_1}\Theta_{e_2\ldots e_m]} + \mathrm{d}K_{[e_1}\wedge u_{e_2\ldots e_m]} \\
&= \frac{1}{m}\varepsilon_{e_1\ldots e_m}\left(C[\boldsymbol{K}] + \omega^{k}{}_{k}\wedge u^{f}K_{f}\right).
\end{aligned}
\tag{3.2.1}
$$

A simple consequence of the definitions and (3.1.2) is

$$
C_{e_1\ldots e_m}[\boldsymbol{K}] + \kappa K_{[e_1}T_{e_2\ldots e_m]} = \mathrm{d}\left(K_{[e_1}u_{e_2\ldots e_m]}\right). \tag{3.2.2}
$$

One can think of equation (3.2.2) as the dual form of (3.1.2), and it is, in fact, equivalent to (3.1.2). We saw in the previous section that the Landau–Lifshitz pseudotensor can be recovered as a pull back of the dual form of Sparling's form along a coordinate section. Thus it might also be worth considering the pull back of $C_{e_1\ldots e_m}[\boldsymbol{K}]$. It is

$$
ms^{*}(C_{e_1\ldots e_m}[\boldsymbol{K}]) = \kappa\varepsilon_{e_1\ldots e_m}\left({}_{\mathrm{LL}}t^{ef}K_{f} + {}_{\mathrm{hE}}\sigma^{eab}\partial_{a}K_{b}\right)s^{*}(\Sigma_{e}). \tag{3.2.3}
$$

Since the Landau–Lifshitz pseudotensor is not a canonical one we cannot *a priori* use the Lagrangian scenario to construct the conserved pseudocurrent. Equation (3.2.3), however, suggests a way of *defining* the gravitational pseudocurrent in the Landau–Lifshitz approach: although ${}_{\mathrm{LL}}t^{\alpha\beta}$ is not canonical, the pseudocurrent is similar to that of the canonical pseudotensors:

$$
\begin{aligned}
{}_{\mathrm{LL}}C^{\mu}[\boldsymbol{K}] &:= {}_{\mathrm{LL}}t^{\mu\nu}K_{\nu} + \left({}_{\mathrm{hE}}\sigma^{\mu[\alpha\beta]} + {}_{\mathrm{hE}}\sigma^{\alpha[\beta\mu]} + {}_{\mathrm{hE}}\sigma^{\beta[\alpha\mu]}\right)\partial_{\alpha}K_{\beta} \\
&= {}_{\mathrm{LL}}t^{\mu\nu}K_{\nu} + {}_{\mathrm{hE}}\sigma^{\mu\alpha\beta}\partial_{\alpha}K_{\beta}.
\end{aligned}
\tag{3.2.4}
$$

One can see that ${}_{\mathrm{hE}}\sigma^{\mu\alpha\beta}$ also plays the role of the spin pseudotensor in the Landau–Lifshitz case.

3.3. The Noether form as the canonical Noether current

The introduction of the Sparling and Noether forms in the previous section is somewhat heuristic as it is based on the fortunate fact that the curvature only appears in the exterior derivative of u_i through the Einstein tensor. Furthermore the fact that they are closely related to the gravitational energy–momentum and the canonical Noether (pseudo) current is justified *a posteriori*, calculating their pull backs. Thus this way of justification is shown by the following diagram:

$$\begin{array}{ccc} \Lambda_{\rm E} & -?\rightarrow & C[K] \\ | & & | \\ \text{pull back } s^* & & \text{pull back } s^* \\ \downarrow & & \downarrow \\ \mathcal{L}_{\rm E} & -\frac{\text{Lagrangian}}{\text{scenario}}\rightarrow & {}_{\rm E}C^{\mu}[K] \end{array}$$

the pull back of $C[K]$ is just the canonical Noether (pseudo) current derived, according to the Lagrangian scenario, from the Lagrangian $\mathcal{L}_{\rm E}$, which is the pull back of $\Lambda_{\rm E}$. One might therefore claim to have a Lagrangian scenario in $L(M)$ to clarify the role of the various differential forms on $L(M)$ and to give a gauge (i.e. local section) independent verification that $C[K]$ is, in fact, the canonical Noether current.

Although the derivation of Einstein's equation on $L(M)$ is not needed in what follows, for the sake of completeness and the beauty of the calculation we first consider it.

Let $g(t)$ be a smooth one-parameter family of metrics of signature $p-q$, $t \in (-\tau, \tau)$ for some $\tau > 0$, such that $g(0) = g$ and $g_{ab}(t)$ be the corresponding functions on $L(M)$. ($g_{ab}(t)$ is a 'one-parameter deformation of g_{ab}'.) If the dot denotes the differentiation with respect to t at $t = 0$, e.g. $\dot{g}_{ab} := ((\mathrm{d}/\mathrm{d}t)g_{ab}(t))_{t=0}$, 'the first variation of g_{ab} determined by the deformation $g_{ab}(t)$', then

$$\dot{\Sigma}_{ab} = \tfrac{1}{2}g^{rs}\dot{g}_{rs}\Sigma_{ab}$$

and

$$\dot{\Omega}^a{}_b = \mathrm{d}\,(\dot{\omega}^a{}_b) + \dot{\omega}^a{}_e \wedge \omega^e{}_b + \omega^a{}_e \wedge \dot{\omega}^e{}_b.$$

Thus, using (1.2.6) and (1.2.11)–(1.2.12), one has

$$2\kappa\dot{\Lambda}_{\rm H} = \left(\Omega^{ab} \wedge \Sigma_{ab}\right)^{\cdot} = -G^{ab}\dot{g}_{ab}\Sigma + \mathrm{d}\Big(\dot{\omega}^a{}_e g^{eb} \wedge \Sigma_{ab}\Big) \tag{3.3.1}$$

and

$$2\kappa\dot{\Lambda}_{\rm E} = -G^{ab}\dot{g}_{ab}\Sigma + \mathrm{d}\Big((\omega^{ea}g^{fb} \wedge \Sigma_{ef} - \tfrac{1}{2}g^{ab}\omega^{ef} \wedge \Sigma_{ef})\dot{g}_{ab}\Big). \tag{3.3.2}$$

Therefore the variation of both $\Lambda_{\rm H}$ and $\Lambda_{\rm E}$ yields the 'Einstein m-form' $G^{ab}\Sigma$ and some exact form. If one wants to recover the field equations on M then a local section $s : U \rightarrow L(M)$ and, in general, a one-parameter family of its deformations $s(t)$ should be considered. For each fixed $p \in U$ $s(p,t)$ is a (not necessariliy non-degenerate) curve in $L(M)$ and hence in a neighbourhood of $s(U)$ in $L(M)$ it defines a vertical vector field V pointwise as the tangents of these curves. Thus if Λ is either $\Lambda_{\rm H}$ or $\Lambda_{\rm E}$ then, using $\mathrm{d}\Lambda = 0$, $(s^*(\Lambda))^{\cdot} = s^*(\dot\Lambda) + s^*(\text{Ł}_V\Lambda) = s^*(\dot\Lambda) + \mathrm{d}s^*(\iota_V\Lambda) = -(1/2\kappa)G^{ab}\dot g_{ab}s^*(\Sigma)+$ exact m-form on U; consequently the functional derivative of the action functional $I_D[s, s^*g] := \int_D s^*(\Lambda)$, $D \subset U$, is $-(1/2\kappa)$ times the Einstein tensor. For coordinate sections one can choose $V = 0$, but for rigid sections V is determined by $\dot g_{ab}$ up to a combination of the fundamental vector fields associated with the elements of $so(p,q) \subset gl(m, \boldsymbol{R})$.

Recall that the canonical Noether current on the spacetime, associated with a vector field $\boldsymbol{K}$ on M, is introduced through the so-called Noether identity. Thus it would be natural to look for the Noether-like identity on $L(M)$ only for the horizontal lift of $\boldsymbol{K}$. However, the calculations can easily be carried out even for a general vector field X on $L(M)$, furthermore a special choice for the vertical component vX of X we will obtain an interesting relation to Komar's superpotential, we work with a general X and specify hX and vX at the final stage of the calculations.

Because of (1.2.16) and $\mathrm{d}\Lambda_{\rm E} = 0$ one has ('Noether identity on $L(M)$ for the Lagrangian $\Lambda_{\rm E}$ and vector field X')

$$\begin{aligned}\tfrac{1}{2}\mathrm{D}(hX)^e \wedge \Omega^{ab} \wedge \Sigma_{eab} &= \kappa \text{Ł}_X \Lambda_{\rm E} - \tfrac{1}{2}\mathrm{d}\Big(\iota_X\Omega^{ab}\wedge\Sigma_{ab} - \text{Ł}_X(\omega^{ab}\wedge\Sigma_{ab})\Big)\\ &= \mathrm{d}\Big(\kappa\iota_X\Lambda_{\rm E} - \tfrac{1}{2}\iota_X\Omega^{ab}\wedge\Sigma_{ab} + \tfrac{1}{2}\text{Ł}_X(\omega^{ab}\wedge\Sigma_{ab})\Big).\end{aligned} \tag{3.3.3}$$

The canonical Noether current, which is, by definition, the $(m-1)$-form in the large brace on the right, is

$$\begin{aligned}\kappa\iota_X\Lambda_{\rm E} &- \tfrac{1}{2}\iota_X\Omega^{ab}\wedge\Sigma_{ab} + \tfrac{1}{2}\iota_X\Big(\Omega^{ab}\wedge\Sigma_{ab} + \omega^a{}_e\wedge\omega^{eb}\wedge\Sigma_{ab}\Big)\\ &\quad + \tfrac{1}{2}\mathrm{d}\Big((vX)^{ab}\Sigma_{ab} - (hX)^e\omega^{ab}\wedge\Sigma_{eab}\Big)\\ &= \tfrac{1}{2}(hX)^e\Omega^{ab}\wedge\Sigma_{eab} + \mathrm{d}\Big((hX)^e u_e\Big) + \tfrac{1}{2}\mathrm{d}\Big((vX)^{ab}\Sigma_{ab}\Big).\end{aligned} \tag{3.3.4}$$

If therefore X is chosen to be the horizontal lift of $\boldsymbol{K}$: $(hX)^e = K^e$ and $(vX) = 0$, then by the Einstein equations and (3.1.2) this is just the Noether form $C[\boldsymbol{K}]$. If, however, $(vX)^a{}_b$ is chosen to be $\nabla_b K^a$ then the last term on the right of (3.3.4) does not vanish and is just Komar's identically conserved horizontal (i.e. tensorial) expression $\frac{1}{2}\nabla_b(\nabla^a K^b - \nabla^b K^a)\Sigma_a$. The canonical Noether current for $\Lambda_{\rm E}$ associated with $X = K^e B(\delta_e) + \nabla_b K^a D_a{}^b$ is therefore the sum of the non-horizontal Noether form, being connected to the horizontal part of X (i.e. to displacements on M), and Komar's horizontal, identically conserved current, being connected to the vertical part of X (i.e. to the element $\nabla_b K^a$ of the Lie algebra $gl(m, \boldsymbol{R})$, defined by the displacement).

A similar analysis for $\Lambda_{\rm H}$ yields

$$\begin{aligned}\tfrac{1}{2}\mathrm{D}(hX)^e\wedge\Omega^{ab}\wedge\Sigma_{eab} &= \kappa\text{Ł}_X\Lambda_{\rm H} - \tfrac{1}{2}\mathrm{d}\Big(\iota_X\Omega^{ab}\wedge\Sigma_{ab}\Big)\\ &= \mathrm{d}\Big(\tfrac{1}{2}(hX)^e\Omega^{ab}\wedge\Sigma_{eab}\Big)\end{aligned}$$

which is nothing but the identity $-\nabla_a K_b G^{ab} = -\nabla_a(G^{ab}K_b)$ in the language of bundle connections.

4. The Belinfante–Rosenfeld equations on $L(M)$

The results of the previous two sections suggest to consider Sparling's form and the Nester–Witten form as the energy–momentum $(m-1)$ form and the corresponding superpotential, being an $(m-2)$ form, on $L(M)$. The canonical Noether current of the classical Lagrangian theory of matter fields is built up not only from the canonical energy–momentum tensor, but the spin tensor too. Similarly, to build up the gravitational canonical Noether (pseudo) current both the canonical energy–momentum and spin (pseudo) tensors are needed. Thus, recalling the structure of the Noether form $C[K]$, the Nester–Witten form plays the role of the spin form too.

However, the most important characteristic feature of the contravariant form of these canonical (pseudo) tensors is the pair of Belinfante–Rosenfeld equations for them, and that their Belinfante–Rosenfeld combination is gauge invariant and independent of total (coordinate) divergences added to the Lagrangian. Thus we may expect to have an exterior differential equation on $L(M)$ whose pull backs are just the Belinfante–Rosenfeld equations on the spacetime.

Such an exterior differential equation can be derived only if the energy–momentum is represented by an m form and the spin by an $(m-1)$ form on $L(M)$. It turns out that the energy–momentum and spin can, in fact, be represented by not only $(m-1)$ and $(m-2)$ forms, respectively, but m and $(m-1)$ forms, respectively, as well. These forms will be studied in the next two sections. Finally we return to the discussion of the Landau–Lifshitz pseudotensors and it will be indicated that the differential geometric formalism implies additional strange properties of the Landau–Lifshitz pseudotensors.

4.1. The energy–momentum m and spin $(m-1)$ forms on $L(M)$

The quantities in the Belinfante–Rosenfeld equations on the spacetime have two free indexes, thus the spin form would have two free indexes too. Since the spin (pseudo) tensors are three index quantities and we would like to recover them as the duals of the components of the pull backs (as in the case of Sparling's form), the spin form must be an $(m-1)$ form. Similarly, the energy–momentum form must be an m-form. Clearly, the spin form must be a linear, while the energy–momentum m-form be a quadratic expression of the connection form.

First consider the spin form, which we define as

$$\begin{aligned} s^j{}_i :&= \tfrac{1}{2}\Big(\delta^j_i \omega^{ab}\wedge\Sigma_{ab} + \omega^{ej}\wedge\Sigma_{ie} + \omega^e{}_i\wedge\Sigma^j{}_e\Big) \\ &= \theta^j\wedge u_i + \tfrac{1}{2}d\Sigma^j{}_i - \tfrac{1}{2}\Xi^c\wedge\Sigma^j{}_{ic}. \end{aligned} \tag{4.1.1}$$

This is a $gl(m,\boldsymbol{R})$-valued pseudotensorial $(m-1)$-form on $L(M)$ of type $adGL(m,\boldsymbol{R})$. (Although $s^j{}_i$ is well defined for $m\geqslant 2$, the second equality holds only if u_i is defined; i.e. if $m\geqslant 3$.) It is interesting that, apart from numerical factors, its trace $s^i{}_i$ is just the $(m-1)$-form $(1/2\kappa)\omega^{ab}\wedge\Sigma_{ab}$ whose exterior derivative has been dropped from $\Lambda_{\rm H}$ to obtain $\Lambda_{\rm E}$. If $\Xi^e=0$ then the pull back of $s^j{}_i$ and (4.1.1) along a coordinate section is

$$s^*(s^j{}_i) = \kappa_{\rm hE}\, s^{ej}{}_i s^*(\Sigma_e) = \frac{1}{2\sqrt{|g|}}\Big(-\mathsf{U}_i{}^{je} + \partial_f\big(\sqrt{|g|}G^{jkef}g_{ki}\big)\Big)s^*(\Sigma_e) \tag{4.1.2}$$

where

$$_{\text{hE}}s^{\mu\alpha}{}_{\beta} := \frac{1}{\sqrt{|g|}}\frac{\partial\mathcal{L}_{\text{hE}}}{\partial\partial_\mu g_{\rho\nu}}(-\delta^\alpha_\rho g_{\nu\beta} - \delta^\alpha_\nu g_{\rho\beta})$$

is the canonical spin pseudotensor for Einstein's Lagrangian $\mathcal{L}_{\text{hE}}$. If s is a rigid section then

$$s^*(s^j{}_i) = \kappa\,{}_{\text{aE}}s^{\mu\alpha}{}_{\beta}\vartheta^j_\alpha E^\beta_i s^*(\Sigma_\mu) = -\tfrac{1}{2}\vee_\beta{}^{\alpha\mu}\vartheta^j_\alpha E^\beta_i s^*(\Sigma_\mu) \tag{4.1.3}$$

where

$$_{\text{aE}}s^{\mu\alpha}{}_{\beta} := \frac{1}{\sqrt{|g|}}\frac{\partial\mathcal{L}_{\text{aE}}}{\partial\partial_\mu\vartheta^\tau_\rho}(-\delta^\alpha_\rho\vartheta^\tau_\beta)$$

is the canonical spin *tensor* in the anholonomic description [8]. Since $_{\text{hE}}s^{\mu\alpha}{}_{\beta}$ is not antisymmetric in μ and α these canonical spin (pseudo) tensors cannot be recovered from a single $\boldsymbol{R}^{m*}$-valued $(m-2)$ form on $L(M)$ as the duals of the pull backs.

For a moment let us consider general, not necessarily torsion-free metric connections. Then using the definitions and the formulae of section 1.2 we have

$$\begin{aligned} -\,\mathrm{d}s^j{}_i &= \tfrac{1}{2}(G^j{}_i + G_i{}^j)\Sigma + \tfrac{1}{2}\Xi^c\wedge\omega^{ab}\wedge\left(\delta^j_i\Sigma_{abc} + \delta^j_b\Sigma_{iac} + g_{bi}\Sigma^j{}_{ac}\right) + t^j{}_i \\ &= -\,\mathrm{d}\theta^j\wedge u_i + \theta^j\wedge\mathrm{d}u_i + \tfrac{1}{2}d\left(\Xi^c\wedge\Sigma^j{}_{ic}\right) \end{aligned} \tag{4.1.4}$$

where

$$\begin{aligned} t^j{}_i &:= -\tfrac{1}{2}(\delta^j_i\omega^a{}_e\wedge\omega^{eb}\wedge\Sigma_{ab} + \omega^a{}_i\wedge(\omega^{bj} - \omega^{jb})\wedge\Sigma_{ab} + \omega^a{}_e\wedge(\omega^{ej} + \omega^{je})\wedge\Sigma_{ia}) \\ &= \theta^j\wedge t_i + \omega^j{}_e\wedge\theta^e\wedge u_i. \end{aligned} \tag{4.1.5}$$

The following theorem shows that the Einstein theory, up to an unspecified cosmological constant, can also be characterized by $t^j{}_i$ and $s^j{}_i$.

Theorem 4.1.1. Let $T_i = T^j{}_i\Sigma_j$ be any horizontal $\boldsymbol{R}^{m*}$-valued $(m-1)$-form on $L(M)$, satisfying $\mathrm{D}T_i := \mathrm{d}T_i - \omega^e{}_i\wedge T_e = 0$ and $\kappa\in\boldsymbol{R}$. If $m\neq 2$ then the following statements are equivalent:

(1) $\omega^a{}_b$ is torsion free, $\Xi^e = 0$, and $\exists\lambda\in\boldsymbol{R}$ such that $G^j{}_i + \lambda\delta^j_i = \kappa T^j{}_i$;
(2) $(\kappa T^j{}_i - \lambda\delta^j_i)\Sigma + t^j{}_i = -\mathrm{d}s^j{}_i$;
(3) $\mathrm{d}(\kappa T^j{}_i\Sigma + t^j{}_i) = 0$.

Proof.

(1) $\to$ (2): If $\Xi^e = 0$ then $G^j{}_i = G_i{}^j$ and because of (4.1.4) statement (2) follows.

(2) $\to$ (3): Since $\mathrm{d}\Sigma = 0$, (3) is a direct consequence of (2).

(3) $\to$ (1): First calculate the exterior derivative of $t^j{}_i$. After a simple but long calculation we arrive at

$$\begin{aligned} -\,2\mathrm{d}t^j{}_i = \Omega^a{}_e\wedge\Big(&\delta^j_i\left(\omega^{eb} - \omega^{be}\right)\wedge\Sigma_{ab} + 2g^{je}\omega^b{}_i\wedge\Sigma_{ab} + \delta^e_i\left(\omega^{bj} - \omega^{jb}\right)\wedge\Sigma_{ab} \\ &+ \left(\omega^{ej} + \omega^{je}\right)\wedge\Sigma_{ia}\Big) + \Xi^e\wedge\Big(\delta^j_i\omega^a{}_c\wedge\omega^{cb}\wedge\Sigma_{abe} \\ &+ \omega^a{}_i\wedge\left(\omega^{bj} - \omega^{jb}\right)\wedge\Sigma_{abe} + \omega^a{}_b\wedge\left(\omega^{bj} + \omega^{jb}\right)\wedge\Sigma_{iae}\Big). \end{aligned}$$

Furthermore, because of the condition imposed on T_i, $d(T^j{}_i\Sigma) = d(\theta^j \wedge T_i) = \Xi^j \wedge T_i - \omega^j{}_e \wedge \theta^e \wedge T_i - \theta^j \wedge dT_i = \Xi^j \wedge T_i - T^e{}_i\omega^j{}_e \wedge \Sigma + T^j{}_e\omega^e{}_i \wedge \Sigma$, thus the condition (3) of the theorem takes the form

$$0 = -2d(\kappa T^j{}_i\Sigma + t^j{}_i) = -2\kappa\Xi^j \wedge T_i + 2\kappa(T^e{}_i\omega^j{}_e - T^j{}_e\omega^e{}_i) \wedge \Sigma - 2dt^j{}_i. \tag{4.1.6}$$

Recall that $\{D_m{}^n, B(\delta_k)\}$ and $\{\omega^a{}_b, \theta^e\}$ are dual, $\omega^a{}_b(D_m{}^n) = \delta^a_m\delta^n_b$ and $\theta^e(D_m{}^n) = 0$. Thus taking the interior product of (4.1.6) first with $D_m{}^n$ and with $D_r{}^s$ and then contracting in j and r and in m and s we obtain: $0 = (2-m)\Xi^e \wedge \Sigma_{ibe}$. For $m \neq 2$ this implies the vanishing of the torsion. Substituting $\Xi^e = 0$ back to (4.1.6) we have

$$0 = 2\Big(\big(R^j{}_e - \kappa T^j{}_e\big)\,\omega^e{}_i - (R^e{}_i - \kappa T^e{}_i)\omega^j{}_e\Big) \wedge \Sigma.$$

Taking its interior product with $D_m{}^n$ and contracting in n and i:

$$R^j{}_m - \frac{1}{m}R\delta^j_m = \kappa\left(T^j{}_m - \frac{1}{m}T^k{}_k\delta^j_m\right).$$

This equation can be rewritten in the following form:

$$G^j{}_i - \kappa T^j{}_i = -\delta^j_i\left(\frac{1}{2m}(m-2)R + \frac{\kappa}{m}T^k{}_k\right) =: -\delta^j_i\,\lambda.$$

But then, because of the contracted (second) Bianchi identity and the differential condition imposed on T_i, λ must be constant. □

Although, as one can show by the same technique, conditions (1) and (2) are equivalent for any *fixed*, e.g. zero, cosmological constant, and they imply condition (3), but the Einstein equation can be recovered from the 'conservation equation' (3) only up to *some*, unspecified cosmological constant. Thus this theorem is a little bit weaker than the theorem of Sparling and Dubois-Violette and Madore. In the rest of this paper $\lambda = 0$ and $\Xi^e = 0$ will be assumed.

One can also calculate the pull backs of condition (2) of theorem 4.1.1, but the technique is the same as that used in section 2, thus only the results will be given. If s is a coordinate section then $s^*(t^j{}_i) = \kappa\,{}_{\rm hE}t^j{}_i s^*(\Sigma)$, while for a rigid section $s^*(t^j{}_i) = (\kappa\,{}_{\rm AE}\theta^\alpha{}_\beta\vartheta^j_\alpha E^\beta_i + \frac{1}{2}\ \mathrm{V}_\beta{}^{\alpha\mu}\ \nabla_\mu(\vartheta^j_\alpha E^\beta_i))s^*(\Sigma)$. Thus $t^j{}_i$ seems to be the energy–momentum m-form on $L(M)$, and the pull backs of condition (2) give the relation between the canonical energy–momentum and spin (pseudo) tensors: $\sqrt{|g|}(T^\alpha{}_\beta + {}_{\rm hE}t^\alpha{}_\beta) = -\partial_\mu(\sqrt{|g|}_{\rm hE}s^{\mu\alpha}{}_\beta)$ for holonomic and $T^\alpha{}_\beta + {}_{\rm AE}\theta^\alpha{}_\beta = -\nabla_{\mu\,{\rm AE}}s^{\mu\alpha}{}_\beta$ for anholonomic sections.

4.2. *The Belinfante–Rosenfeld equations*

First define

$$\begin{aligned}\sigma^j{}_i &:= s^j{}_i - \tfrac{1}{2}d\Sigma^j{}_i = \tfrac{1}{2}\omega^{ab} \wedge \left(\delta^j_i\Sigma_{ab} + \delta^j_a\Sigma_{bi} - \delta^j_b\Sigma_{ai}\right) \\ &= \theta^j \wedge u_i\end{aligned} \tag{4.2.1}$$

and recall that in (4.1.2) and (4.1.3) the pull backs $s^*(s^j{}_i - \frac{1}{2}\mathrm{d}\Sigma^j{}_i)$ were just the von Freud and Møller–Goldberg superpotentials, respectively. But, as we noted in section 3.1, they are the contravariant forms of the canonical spin (pseudo) tensors, thus $s^*(\sigma^j{}_i) = \kappa_{\mathrm{hE}}\sigma^{ej}{}_i s^*(\sigma_e)$ for holonomic and $s^*(\sigma^j{}_i) = \kappa\,{}_{\mathrm{AE}}\sigma^{\mu\alpha}{}_\beta \vartheta^j_\alpha E^\beta_i s(\Sigma_\mu)$ for anholonomic sections. Moreover, in the pull back (3.1.3) of the Noether form $C[K]$ along a general section $\theta^a \wedge u^b$ appeared as the spin term. Thus it is σ^{ji} that should be considered as the contravariant form of the spin form. Its exterior derivative is

$$-\,\mathrm{d}\sigma^{ji} = \kappa T^{ji}\Sigma + \Theta^{ji} \tag{4.2.2}$$

where

$$\Theta^{ji} := t^{ji} + (\omega^{ie} + \omega^{ei}) \wedge \sigma^j{}_e. \tag{4.2.3}$$

The pull back of Θ^{ji} along a coordinate section is $s^*(\Theta^{ji}) = \kappa\,{}_{\mathrm{hE}}\theta^{ji}s^*(\Sigma)$, i.e. it gives the contravariant form of Einstein's canonical energy–momentum (i.e. Bergmann's) pseudotensor, while for a rigid section it deviates from ${}_{\mathrm{AE}}\theta^{\alpha\beta}$. However, the pull backs of the full equation (4.2.2) are just the relations between the contravariant form of the canonical (pseudo) tensors: $-\partial_\mu(\sqrt{|g|}\,{}_{\mathrm{hE}}\sigma^{\mu\alpha\beta}) = \sqrt{|g|}(T^{\alpha\beta} + {}_{\mathrm{hE}}\theta^{\alpha\beta})$ if s is a coordinate section, and $-\nabla_\mu\,{}_{\mathrm{AE}}\sigma^{\mu\alpha\beta} = T^{\alpha\beta} + {}_{\mathrm{AE}}\theta^{\alpha\beta}$ if s is a rigid section. Thus the antisymmetric part of (4.2.2) seems to be the differential geometric form of the algebraic Belinfante–Rosenfeld equation. Since $4\sigma^{[ji]}(D^{mn}) = 3(g^{j[i}\Sigma^{mn]} - g^{i[j}\Sigma^{mn]})$, $\sigma^{[ji]}$ and $\Theta^{[ji]}$ vanish in two dimensions. If $m \geqslant 3$ then, however, $\Theta^{[ji]}$ is never zero, since $\Theta^{[ji]} = 0$ would imply the contradiction $(m-2)\Sigma_{ab} = 0$, and hence $\sigma^{[ji]}$ is not a closed form.

The Belinfante–Rosenfeld combination of the contravariant form of the canonical (pseudo) tensors is Einstein's tensor [8]:

$$\sqrt{|g|}\,{}_{\mathrm{hE}}\theta^{\alpha\beta} + \partial_\mu\Big(\sqrt{|g|}({}_{\mathrm{hE}}\sigma^{\mu[\alpha\beta]} + {}_{\mathrm{hE}}\sigma^{\alpha[\beta\mu]} + {}_{\mathrm{hE}}\sigma^{\beta[\alpha\mu]})\Big) = -\frac{1}{\kappa}\sqrt{|g|}G^{\alpha\beta}$$

$${}_{\mathrm{AE}}\theta^{\alpha\beta} + \nabla_\mu\Big({}_{\mathrm{AE}}\sigma^{\mu[\alpha\beta]} + {}_{\mathrm{AE}}\sigma^{\alpha[\beta\mu]} + {}_{\mathrm{AE}}\sigma^{\beta[\alpha\mu]}\Big) = -\frac{1}{\kappa}G^{\alpha\beta}$$

being spacetime gauge invariant and independent of total (coordinate) divergences added to the Lagrangian. Thus rewriting (4.2.2) in the form $\Theta^{ji} + \mathrm{d}\sigma^{ji} = -G^{ji}\Sigma$ and recalling the antisymmetry ${}_{\mathrm{hE}}\sigma^{\mu\alpha\beta} = {}_{\mathrm{hE}}\sigma^{[\mu\alpha]\beta}$ and ${}_{\mathrm{AE}}\sigma^{\mu\alpha\beta} = {}_{\mathrm{AE}}\sigma^{[\mu\alpha]\beta}$ (4.2.2) can also be considered as the differential geometric form of the Belinfante–Rosenfeld combination of the canonical pseudotensors ${}_{\mathrm{hE}}\theta^{\alpha\beta}$ and ${}_{\mathrm{hE}}\sigma^{\mu\alpha\beta}$ and of the canonical tensors ${}_{\mathrm{AE}}\theta^{\alpha\beta}$ and ${}_{\mathrm{AE}}\sigma^{\mu\alpha\beta}$.

To obtain a 'global conservation equation' now a two-index tensor field with the corresponding function K_{ab} on $L(M)$ and an m-dimensional submanifold D with boundary should be involved instead of a vector field K_a and an $(m-1)$-dimensional submanifold. Now

$$-\,\mathrm{d}\Big(\sigma^{ab}K_{ab}\Big) = -\mathrm{d}K_{ab} \wedge \sigma^{ab} + K_{ab}\Big(\Theta^{ab} + \kappa T^{ab}\Sigma\Big)$$

which is especially interesting for an antisymmetric K_{ab} generating a coordinate rotation. It describes how the (non-conserved) total spin of gravity varies as one passes from an $(m-1)$-dimensional submanifold to another being homologous to the previous one via D.

4.3. More on the Landau–Lifshitz pseudotensors

For the sake of completeness, finally, let us consider the dual form of the spin form:

$$\sigma^j{}_{e_2\ldots e_m} := \sigma^{je}\varepsilon_{ee_2\ldots e_m}. \tag{4.3.1}$$

Then

$$-\,\mathrm{d}\sigma^j{}_{e_2\ldots e_m} = \kappa T^j{}_{e_2\ldots e_m}\Sigma + \Theta^j{}_{e_2\ldots e_m} \tag{4.3.2}$$

where $T^j{}_{e_2\ldots e_m} := T^{je}\varepsilon_{ee_2\ldots e_m}$ and

$$\Theta^j{}_{e_2\ldots e_m} := \Theta^{je}\varepsilon_{ee_2\ldots e_m} - \omega^k{}_k \wedge \sigma^{je}\varepsilon_{ee_2\ldots e_m}. \tag{4.3.3}$$

The pull back of $\sigma^j{}_{e_2\ldots e_m}$ along a coordinate section is

$$s^*(\sigma^j{}_{e_2\ldots e_m}) = -\frac{1}{2}\frac{1}{(m-1)!}\partial_f\left(|g|G^{jref}\right)\epsilon_{ee_2\ldots e_m}\epsilon_{rr_2\ldots r_m}\mathrm{d}x^{r_2}\wedge\ldots\wedge\mathrm{d}x^{r_m} \tag{4.3.4}$$

and hence

$$-\,s^*(\mathrm{d}\sigma^j{}_{e_2\ldots e_m}) = \frac{1}{2}\frac{1}{m!}\partial_r\partial_s\left(|g|G^{jres}\right)\epsilon_{ee_2\ldots e_m}\epsilon_{l_1\ldots l_m}\mathrm{d}x^{l_1}\wedge\ldots\wedge\mathrm{d}x^{l_m}. \tag{4.3.5}$$

Therefore the pull back $s^*(\Theta^j{}_{e_2\ldots e_m})$ must yield the Landau–Lifshitz pseudotensor again. In fact, it is

$$s^*(\Theta^j{}_{e_2\ldots e_m}) = \kappa\,{}_{\mathrm{LL}}t^{je}\varepsilon_{ee_2\ldots e_m}s^*(\Sigma) \tag{4.3.6}$$

and therefore the pull back of (4.3.2) along a coordinate section is equivalent to

$$-\,\kappa\partial_\mu(|g|\,{}_{\mathrm{hE}}\sigma^{\mu\alpha\beta}) = |g|\left(G^{\alpha\beta} + \kappa\,{}_{\mathrm{LL}}t^{\alpha\beta}\right). \tag{4.3.7}$$

Thus ${}_{\mathrm{LL}}\sigma^{\mu\alpha\beta} := {}_{\mathrm{hE}}\sigma^{\mu\alpha\beta}$ also plays the role of the spin pseudotensor in the Landau–Lifshitz approach too, in accordance with the results of section 3.2. The symmetry of $G^{\alpha\beta}$ and ${}_{\mathrm{LL}}t^{\alpha\beta}$ implies $\partial_\mu(|g|_{\mathrm{LL}}\sigma^{\mu[\alpha\beta]}) = 0$, and hence the Belinfante–Rosenfeld combination of ${}_{\mathrm{LL}}t^{\alpha\beta}$ and ${}_{\mathrm{LL}}\sigma^{\mu\alpha\beta}$ is also tensorial:

$$\begin{aligned}|g|_{\mathrm{LL}}t^{\alpha\beta} &+ \partial_\mu\left(|g|({}_{\mathrm{LL}}\sigma^{\mu[\alpha\beta]} + {}_{\mathrm{LL}}\sigma^{\alpha[\beta\mu]} + {}_{\mathrm{LL}}\sigma^{\beta[\alpha\mu]})\right)\\ &= |g|_{\mathrm{LL}}t^{\alpha\beta} + \partial_\mu(|g|_{\mathrm{LL}}\sigma^{\mu(\alpha\beta)}) = -\frac{1}{\kappa}|g|G^{\alpha\beta}.\end{aligned}$$

The Landau–Lifshitz pseudocurrent (3.2.4) can be derived from a superpotential:

$${}_{\mathrm{LL}}C^\mu[K] = -\frac{1}{\kappa}G^{\mu\nu}K_\nu + \frac{1}{2\kappa|g|}\partial_\alpha\left(\partial_\beta(|g|G^{\alpha\mu\beta\nu})K_\nu\right).$$

Thus $|g|(\kappa_{\mathrm{LL}}C^\mu[K] + G^{\mu\nu}K_\nu)$ is always pseudoconserved, in contrast to $|g|(\kappa_{\mathrm{LL}}t^{\mu\nu} + G^{\mu\nu})K_\nu$ considered in section 2.3. However, if K is a Killing vector of the geometry then, in general, it is *not* the sum of separately (pseudo) conserved gravitational and matter pseudocurrents either, which could be expected on physical grounds (see the introduction). We have only

$$0 = \partial_\mu \Big(|g|(\kappa_{\rm LL} C^\mu[K] + G^{\mu\nu} K_\nu) \Big)$$
$$= |g| \tfrac{1}{2} G^{\mu\nu} \mathcal{L}_K g_{\mu\nu} + \kappa \partial_\mu (|g|_{\rm LL} C^\mu[K]) + |g| G^{\mu\nu} K_\nu \Gamma^\rho_{\mu\rho}$$

and hence the Landau–Lifshitz pseudocurrent is *not* pseudoconserved even for a Killing vector K. For vector fields satisfying $\partial_{(\alpha} K_{\beta)} = 0$ the pseudocurrent $|g|(\kappa_{\rm LL} C^\mu[K] + G^{\mu\nu} K_\nu)$ is the sum of two separately pseudoconserved parts: the first, as we saw in section 2.3, is $|g|(\kappa_{\rm LL} t^{\mu\nu} + G^{\mu\nu}) K_\nu$; and the second is $|g|_{\rm LL} \sigma^{\mu\alpha\beta} \partial_\alpha K_\beta$. However, if K_ν generates coordinate rotation then the second part is not zero. But accepting $\sigma^j{}_i$ to be the geometric object describing the spin distribution of gravity, which interpretation is suggested by the results of the previous sections and that the relation between ${}_{\rm LL}\sigma^{\mu\alpha\beta}$ and ${}_{\rm LL}t^{\alpha\beta}$ is the same that between e.g. ${}_{\rm hE}\sigma^{\mu\alpha\beta}$ and ${}_{\rm hE}\theta^{\alpha\beta}$, the second term should be interpreted as the spin angular momentum. The spin and orbital angular momenta of gravity in the Landau–Lifshitz approach are therefore separately conserved. This strange behaviour, together with others mentioned above and at the end of section 2.3 may suggest to consider the Landau–Lifshitz pseudotensors aphysical, in contrast to the canonical (pseudo) tensors.

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