Poisson-Lie generalization of the Kazhdan-Kostant-Sternberg reduction

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- I shall first explain that the trigonometric Ruijsenaars-Schneider models are symplectic reductions of certain Poisson-Lie symmetric 'free systems' on the Heisenberg double of $U(n)$. This generalizes the standard `[KKS 78]` treatment of the trigonometric Sutherland models as reductions of the free particle moving on the group $U(n)$.

- Then, maybe in less detail, I shall discuss how the Ruijsenaars duals of these models arise in the same reduction procedure ('from another set of canonical integrable systems, in another gauge').

Earlier works treating trigonometric Ruijsenaars-Schneider type models in the symmetry reduction framework include [Gorsky-Nekrasov 95], [Fock-Rosly 99], [Fock-Gorsky-Nekrasov-Rubtsov 2000]. These papers inspired us, but none of them derived the trigonometric model with real coordinates using compact Poisson-Lie symmetry, which we consider as the true analogue of [KKS 78].

The integrable many-body models of interest

The trigonometric Ruijsenaars-Schneider (86) model:

$$H_{\mathrm{RS}} = \sum_{k=1}^{n} (\cosh p_k) \Pi_{j\neq k} \left[1 + \frac{\sinh^2 \frac{x}{2}}{\sin^2(q_k - q_j)} \right]^{\frac{1}{2}}$$

It is a relativistic generalization (here with $c = 1$) of

$$H_{\mathrm{Sutherland}} = \frac{1}{2} \sum_{k=1}^{n} p_k^2 + \frac{x^2}{8} \sum_{j\neq k} \frac{1}{\sin^2(q_k - q_j)}$$

The dual models (introduced by Ruijsenaars (88,95)):

$$\widehat{H}_{\mathrm{RS}} = \sum_{k=1}^{n} (\cos \hat{q}_k) \prod_{j\neq k} \left[1 - \frac{\sinh^2 \frac{x}{2}}{\sinh^2(\hat{p}_k - \hat{p}_j)} \right]^{\frac{1}{2}}$$

$$\widehat{H}_{\mathrm{Sutherland}} = \sum_{k=1}^{n} (\cos \hat{q}_k) \prod_{j\neq k} \left[1 - \frac{(x/2)^2}{(\hat{p}_k - \hat{p}_j)^2} \right]^{\frac{1}{2}}$$

The Kazhdan-Kostant-Sternberg reduction

Consider cotangent bundle $T^*U(n)$ of $U(n)$ (in right-trivialization):

$$T^*U(n) = \{(g, J_L) \,|\, g \in U(n),\, J_L \in u(n)^* \simeq u(n)\}$$

It carries the natural symplectic form

$$\Omega(g, J_L) = d\,\mathrm{tr}\,(J_L dg g^{-1})$$

and two sets of 'canonical commuting Hamiltonians' $\{h_k\}$ and $\{\widehat{h}_{\pm k}\}$

$$h_k(g, J_L) := \mathrm{tr}\,(\mathrm{i}J_L)^k, \quad \widehat{h}_k(g, J_L) := \Re\mathrm{tr}\,(g^k), \quad \widehat{h}_{-k}(g, J_L) := \Im\mathrm{tr}\,(g^k)$$

- One can write down their Hamiltonian flows explicitly.

- They are invariant under the adjoint action of $U(n)$ on $T^*U(n)$.

Interesting models are reductions of 'obviously integrable' systems.

The adjoint action of $U(n)$ on the phase space

$$\eta \triangleright (g, J_L) = (\eta g \eta^{-1}, \eta J_L \eta^{-1}) \qquad \forall \eta \in U(n)$$

is generated by the moment map $J : T^*U(n) \to u(n)^*$ given by

$$J(g, J_L) = J_L + J_R \quad \text{with} \quad J_R(g, J_L) := -g^{-1} J_L g.$$

J is sum of moment maps generating left/right multiplication.

With arbitrary real $x \neq 0$, define $\iota(x) \in u(n)$ by

$$\iota(x)_{jj} = 0, \quad \forall j, \qquad \iota(x)_{jk} = \mathrm{i}x, \quad \forall j \neq k.$$

KKS [78] showed that the moment map constraint

$$J = \iota(x)$$

produces the trigonometric Sutherland model from the Hamiltonian system describing the free particle on $U(n)$: $(T^*U(n), \Omega, h_2)$. The Hamiltonians $\{h_k\}$ give action variables of Sutherland model (and the $\{\hat{h}_{\pm k}\}$ become in effect the Sutherland coordinate variables).

Nekrasov [99] and co-workers noticed that $\{\hat{h}_{\pm k}\}$ yields commuting Hamiltonians of the Ruijsenaars dual of the Sutherland model (and the $\{h_k\}$ become in effect the dual coordinate variables).

Poisson-Lie analogue of KKS reduction?

Due to Semenov-Tian-Shansky (1985):

- P-L analogue of $T^*U(n)$ is Heisenberg double of Poisson $U(n)$.

- The Heisenberg double has ‘canonical commuting Hamiltonians’.

As described explicitly by Klimčík [06]:

- Adjoint action (moment map) generalizes to quasi-adjoint action.

We asked:

- **What is the correct analogue of the KKS constant $\iota(x)$?**

If this is known, the rest is in principle straightforward calculation.

Symplectic structure of Heisenberg double

The Heisenberg double of $U(n)$ is the *real* manifold $GL(n,\mathbb{C})$.

Every $K \in GL(n,\mathbb{C})$ admits two Iwasawa decompositions:

$$K = b_L g_R^{-1} \quad \text{and} \quad K = g_L b_R^{-1} \quad \text{with} \quad g_{L,R} \in U(n),\ b_{L,R} \in B$$

B: group of upper triangular matrices with positive diagonal entries.

Define maps $\Lambda_{L,R} : GL(n,\mathbb{C}) \to B$ and $\Xi_{L,R} : GL(n,\mathbb{C}) \to U(n)$ by

$$\Lambda_{L,R}(K) := b_{L,R} \quad \text{and} \quad \Xi_{L,R}(K) := g_{L,R}$$

Due to Semenov-Tian-Shansky (85) and Alekseev-Malkin (94),
$GL(n,\mathbb{C})$ has the natural symplectic form:

$$\omega_+ = \frac{1}{2}\Im\mathrm{tr}\,(d\Lambda_L\Lambda_L^{-1} \wedge d\Xi_L\Xi_L^{-1}) + \frac{1}{2}\Im\mathrm{tr}\,(d\Lambda_R\Lambda_R^{-1} \wedge d\Xi_R\Xi_R^{-1}).$$

The Poisson bracket on $(GL(n,\mathbb{C}),\omega_+)$

For any $\Phi_1,\Phi_2 \in C^\infty(GL(n,\mathbb{C}))$:

$$\{\Phi_1,\Phi_2\}_+ = \Im\mathrm{tr}\left(\nabla^R\Phi_1\rho(\nabla^R\Phi_2)+\nabla^L\Phi_1\rho(\nabla^L\Phi_2)\right)$$

where $\rho := \frac{1}{2}(\pi_{u(n)}-\pi_{\mathcal{B}})$ belongs to the splitting $gl(n,\mathbb{C}) = u(n)+\mathcal{B}$ and we use the $gl(n,\mathbb{C})$-valued derivatives

$$\frac{d}{ds}\Big|_{s=0}\Phi(e^{sX}Ke^{sY}) = \Im\mathrm{tr}\,(X\nabla^L\Phi(K)+Y\nabla^R\Phi(K)) \quad \forall X,Y\in gl(n,\mathbb{C})$$

Iwasawa maps $\Xi_{L,R}: GL(n,\mathbb{C})\to U(n)$ and $\Lambda_{L,R}: GL(n,\mathbb{C})\to B$ are **Poisson maps** if $U(n)$ and B are equipped with their standard Poisson structures. In particular, $\{\ ,\ \}_+$ closes on

$$\Xi^*_{L,R}C^\infty(U(n)) \quad \text{and on} \quad \Lambda^*_{L,R}C^\infty(B)$$

The induced Poisson bracket on $U(n)$ is Sklyanin bracket defined by the r-matrix, $R^{\mathrm{i}}\in\mathrm{End}(u(n))$, $R^{\mathrm{i}}(X)=\pi_{u(n)}(-\mathrm{i}X)$ for $X\in u(n)$.

The dual Poisson-Lie groups $U(n)$ and B

For $f \in C^\infty(B)$ define $d^{L,R}f \in C^\infty(B, u(n))$ by

$$\left.\frac{d}{ds}\right|_{s=0} f(e^{sX} b e^{sY}) = \Im\mathrm{tr}\,\Big(X d^L f(b) + Y d^R f(b)\Big) \qquad \forall X, Y \in \mathcal{B}$$

For $\phi \in C^\infty(U(n))$ define $\mathbf{D}^{L,R}\phi \in C^\infty(U(n), u(n))$ by

$$\left.\frac{d}{ds}\right|_{s=0} \phi(e^{sX} g e^{sY}) = \mathrm{tr}\,\Big(X\mathbf{D}^L\phi(g) + Y\mathbf{D}^R\phi(g)\Big) \qquad \forall X, Y \in \mathcal{G}$$

Explicit form of induced Poisson bracket on B and on $U(n)$ reads

$$\{f_1, f_2\}_B(b) = -\Im\mathrm{tr}\,\Big(b^{-1}(d^L f_1(b)) b\, d^R f_2(b)\Big)$$

$$\{\phi_1, \phi_2\}_{U(n)}(g) = \mathrm{tr}\,\Big(\mathbf{D}^R\phi_1(g) R^{\mathsf{i}}(\mathbf{D}^R\phi_2(g)) - \mathbf{D}^L\phi_1(g) R^{\mathsf{i}}(\mathbf{D}^L\phi_2(g))\Big)$$

Denote $C^\infty(U(n))^{U(n)}$ the adjoint (conjugation) invariant functions and $C^\infty(B)^c$ the center of $\{\ ,\ \}_B$ (the dressing invariants). Then

$$\Lambda_L^* C^\infty(B)^c = \Lambda_R^* C^\infty(B)^c \quad \text{and} \quad \Xi_R^* C^\infty(U(n))^{U(n)}$$

form **Abelian subalgebras** in $C^\infty(GL(n,\mathbb{C}))$ w.r.t. $\{\ ,\ \}_+$.

Quasi-adjoint action

$\Lambda_{L,R} : GL(n,\mathbb{C}) \to B$ are **Poisson-Lie analogues** of moment maps $J_{L,R} : T^*U(n) \to u(n)^*$ generating left/right multiplications.

The product $\Lambda := \Lambda_L \Lambda_R : GL(n,\mathbb{C}) \to B$ is also P-L moment map, which is the analogue of $J_L + J_R$.

It generates the 'infinitesimal quasi-adjoint action.'
For $Y \in u(n)$, $K \in GL(n,\mathbb{C})$ and every function f on $GL(n,\mathbb{C})$,

$$\frac{d}{ds} f(e^{sY} \triangleright K)|_{s=0} := \Im \mathrm{tr}\left(Y\{f,\Lambda\}_+(K)\Lambda(K)^{-1}\right)$$

Klimčík [06] integrated this to $U(n)$ action on $GL(n,\mathbb{C})$:

$$\eta \triangleright K := \eta K \Xi_R(\eta \Lambda_L(K)), \qquad \eta \in U(n), \quad K \in GL(n,\mathbb{C})$$

Now can reduce $(GL(n,\mathbb{C}), \omega_+)$ by choosing any $\nu \in B$ and imposing

$$\text{moment map constraint:} \quad \Lambda(K) = \nu, \quad K \in GL(n,\mathbb{C}).$$

But what dynamics to reduce, and **how to choose** ν?

Poisson-Lie symmetric free motion on $U(n)$

Introduce Hermitian matrix-valued function L on $GL(n,\mathbb{C})$ as

$$L(K) := (K^\dagger K)^{-1} = \Lambda_R(K)\Lambda_R(K)^\dagger, \quad K \in GL(n,\mathbb{C}).$$

For any sequence of real parameters μ_j, define the Hamiltonian

$$H_\mu(K) := \frac{1}{2}\sum_{j\neq 0} \frac{\mu_j}{j}\mathrm{tr}\,(L(K)^j), \quad K \in GL(n,\mathbb{C})$$

- H_μ is invariant w.r.t. quasi-adjoint action of $U(n)$ on $GL(n,\mathbb{C})$.
- Its flow (S. Zakrzewski 97) with initial condition $K = b\eta^{-1}$ reads

$$K_\mu(t) = b\,\mathrm{exp}\left(-\mathrm{i}t\sum_{j\neq 0}\mu_j(b^\dagger b)^{-j}\right)\eta^{-1}$$

'Momentum' Λ_L is constant, 'position' g_R follows geodesic on $U(n)$.

- Elements of $\{H_\mu\}$ commute, and generalize family $\{h_k\}$ on $T^*U(n)$.
- Set $\{\widehat{h}_k\}$ has P-L analogue, too (replace 'coordinate' g by g_R).

Two families of ‘canonical integrable systems’

- First, the Hamiltonian $H = \Lambda_L^* f$ with $f \in \mathbf{C^\infty(B)^c}$ has the flow

$$K(t) = b_L(t) g_R^{-1}(t) = b_L(0) \exp\left[t d^R f(b_L(0))\right] g_R^{-1}(0)$$

In other words, $b_L(t) = b_L(0)$ and $g_R(t) = g_R(0) \exp\left[-t d^R f(b_L(0))\right]$

- Second, the flow of $\hat{H} = \Xi_R^* \phi$ with $\phi \in \mathbf{C^\infty(U(n))^{U(n)}}$ reads

$$g_R(t) = \gamma(t) g_R(0) \gamma(t)^{-1}, \qquad b_L(t) = b_L(0)\beta(t)$$

with $\gamma(t) \in U(n)$, $\beta(t) \in B$ defined by $e^{\mathrm{i}t\mathbf{D}\phi(g_R(0))} = \beta(t)\gamma(t)$. Also

$$K(t)K^\dagger(t) = b_L(t) b_L(t)^\dagger = b_L(0) e^{2\mathrm{i}t\mathbf{D}\phi(g_R(0))} b_L(0)^\dagger$$

Thus solutions are obtained by Gram-Schmidt orthogonalization.

Let P be the space of Hermitian positive definite matrices. Consider the **diffeomorphism** $\Upsilon : B \to P$ defined by $\Upsilon(b) = bb^\dagger$. Υ is $U(n)$-**equivariant**: $\Upsilon(\mathrm{Dress}_g b) = g\Upsilon(b)g^{-1}$, where $\mathrm{Dress}_g(b) := \Lambda_L(gb)$.

$C^\infty(B)^c = C^\infty(B)^{U(n)} = C^\infty(P)^{U(n)}$ adjoint invariants on P

Poisson-Lie analogue of KKS reduction

P-L analogue of $\iota(x)$ is the upper-triangular $n \times n$ matrix $\nu(x)$:

$$\nu(x)_{jj} = 1 \quad \forall j, \qquad \nu(x)_{jk} = (1 - e^{-x}) e^{\frac{(k-j)x}{2}} \quad \forall j < k.$$

Theorem: *Denote by $\mathcal{C}$ the set of the regular elements of a Weyl alcove in the maximal torus $\mathbb{T}_n \subset U(n)$, by A the diagonal subgroup of B, by N the group of complex upper-triangular matrices having 1 all along the diagonal, and by G_x the isotropy group of $\nu(x)$, i.e.,*

$$G_x := \{ g \in U(n) \,|\, g\nu(x)\nu(x)^\dagger g^{-1} = \nu(x)\nu(x)^\dagger \}.$$

Then every solution K of the moment map constraint has the form

$$K = g \triangleright (\mathcal{N}(T) a T^{-1})$$

where $T \in \mathcal{C}$, $a \in A$, $g \in G_x$ and $\mathcal{N}(T) \in N$ is given by

$$\mathcal{N}(T)_{kl} = \prod_{m=1}^{l-k} \frac{e^{\frac{x}{2}} T_l - e^{-\frac{x}{2}} T_{k+m}}{T_l - T_{k+m-1}}, \qquad \forall k < l.$$

No two different points of the form $\mathcal{N}(T) a T^{-1}$ can be transformed into each other by the action of G_x.

In principle identification of reduced system

The theorem says that we have the global ‘gauge slice’ S:

$$S := \{\mathcal{N}(T)aT^{-1} \,|\, T \in \mathcal{C}, a \in A\} \subset \Lambda^{-1}(\nu(x)) \subset GL(n,\mathbb{C}).$$

The ‘position’ variable g_R is diagonal on S. **We found $\nu(x)$ by requiring this to be possible and the maximality of** $\dim(G_x)$**.**

S, equipped with pull-back of ω_+, is model of reduced phase space.

The reduced Hamiltonians are

$$H_\mu(T,a) = \frac{1}{2}\sum_{j\neq 0}\frac{\mu_j}{j}\mathrm{tr}\left(L(T,a)^j\right)$$

with the naturally induced Lax matrix

$$L(T,a) = a^{-1}\mathcal{N}(T)^{-1}(\mathcal{N}(T)^\dagger)^{-1}a^{-1}$$

Concrete identification of reduced system

Choose the following parametrization of T and a:

$$T := \mathrm{diag}(e^{2\mathrm{i}q_1}, e^{2\mathrm{i}q_2}, \ldots, e^{2\mathrm{i}q_n}), \quad 0 \leq q_k < \pi, \quad q_1 > q_2 > \cdots > q_n$$

$$a := \mathrm{diag}(e^{\zeta_1}, e^{\zeta_2}, \ldots, e^{\zeta_n}) \quad \text{where}$$

$$\zeta_k = -\frac{p_k}{2} - \frac{1}{4}\sum_{m<k} \ln\left(1 + \frac{\sinh^2\frac{x}{2}}{\sin^2(q_k - q_m)}\right) + \frac{1}{4}\sum_{m>k} \ln\left(1 + \frac{\sinh^2\frac{x}{2}}{\sin^2(q_k - q_m)}\right)$$

Reduced symplectic form is $\omega_r = \Im\mathrm{tr}\,(T^{-1}dT \wedge a^{-1}da) = \sum_k dp_k \wedge dq_k$.
Thus the reduced phase space is just the cotangent bundle $T^*\mathcal{C}$.

$L(T,a)$ gives rise to the standard Ruijsenaars-Schneider Lax matrix:

$$\mathbf{L}_{kl} = \frac{e^{\frac{p_k+p_l}{2}}\sinh\frac{x}{2}}{\sinh(\frac{x}{2} + \mathrm{i}q_k - \mathrm{i}q_l)} \Pi_{m\neq k}\left[1 + \frac{\sinh^2\frac{x}{2}}{\sin^2(q_k - q_m)}\right]^{\frac{1}{4}} \Pi_{s\neq l}\left[1 + \frac{\sinh^2\frac{x}{2}}{\sin^2(q_l - q_s)}\right]^{\frac{1}{4}}$$

By projecting free flows, we get RS integration algorithm for $H_\mu(q,p)$:

$$T(t) = e^{2\mathrm{i}q(t)} = \mathcal{E}[T(0)e^{\mathrm{i}t\sum_{j\neq 0}\mu_j(\mathbf{L}(0))^j}],$$

where $\mathcal{E}$ transforms into Weyl alcove $\mathcal{C}$. For $\mu_j = \delta_{1,j^2}$, $H_\mu \equiv H_{\mathrm{RS}}$.

On the dual models: general remarks

- ‘Good gauge slice’ S was obtained by diagonalizing the variable $g_R \in U(n)$ by means of the action of the isotropy group G_x.

- Since under the quasi-adjoint action by $\eta \in U(n)$ $b_L b_L^\dagger$ transforms into $\eta(b_L b_L^\dagger)\eta^{-1}$, it is natural to think about an alternative gauge where b_L is diagonal. However, $b_L b_L^\dagger$ cannot be diagonalized by G_x.

- Difficulty evaporates if we use the ‘shifting trick’ of symplectic reduction: extend the phase space by (dressing) orbit and reduce at the identity value of the extended moment map. Then the isotropy group is the full $U(n)$, and ‘B-diagonal gauge’ is available.

- In a rather natural parametrization of the B-diagonal gauge the commuting Hamiltonians $\{\widehat{h}_{\pm k}\}$ turn into another RS type model, which Ruijsenaars `[CMP88, RIMS95]` called the dual model.

- Ruijsenaars was led to duality by construction of action-angle map via diagonalization of the Lax matrix. We recover this from ‘geometric democracy between P-L group and its dual’.

Shifting trick and B-diagonal gauge

Take dressing orbit $\mathcal{O}$ of $U(n)$ (symplectic leaf in B) through $\nu(x)^{-1}$.

Introduce extended phase space and moment map $\Lambda^{\text{ext}}: D^{\text{ext}} \to B$

$$D^{\text{ext}} := GL(n,\mathbb{C}) \times \mathcal{O}, \quad \omega^{\text{ext}} := \omega_+ + \omega_{\mathcal{O}}, \quad \Lambda^{\text{ext}}(K,\beta) := \Lambda(K)\beta.$$

Λ^{ext} generates twisted quasi-adjoint action of $U(n)$. Reduce at e_B.
Hamiltonians of interest do not depend on orbital variable β.

By imposing the ‘partial gauge fixing condition’

$$b_L = \Lambda_L(K) := e^{-\hat{p}}, \quad \hat{p} := \text{diag}(\hat{p}_1, \dots, \hat{p}_n) \quad \hat{p}_1 \geq \hat{p}_2 \geq \cdots \geq \hat{p}_n$$

the moment map constraint becomes

$$e^{-\hat{p}} g_R e^{2\hat{p}} g_R^{-1} e^{-\hat{p}} = \beta^{-1}(\beta^{-1})^\dagger = \left[e^{-x} \frac{e^{xn}-1}{n} v v^\dagger + e^{-x} \mathbf{1}_n \right]$$

with some $v \in \mathbb{C}^n$, $v^\dagger v = n$. This can be solved iff

$$\hat{p}_k - \hat{p}_{k+1} \geq \frac{|x|}{2} \qquad \forall k = 1, \dots, n-1.$$

Hence $\hat{p}$ is regular and residual gauge transformations belong to $\mathbb{T}_n$.

$$\mathcal{C}_{j,k}(\hat{p},x) := \frac{e^{-\hat{p}_j}e^{-\hat{p}_k}\sinh\frac{x}{2}}{\sinh\left(\hat{p}_j-\hat{p}_k+\frac{x}{2}\right)}, \qquad \mathcal{R}_j(\hat{p},x) := e^{2\hat{p}_j}\prod_{m\neq j}\frac{\sinh(\hat{p}_j-\hat{p}_m-\frac{x}{2})}{\sinh(\hat{p}_j-\hat{p}_m)}$$

$$Q(\hat{p})_{j,k} := \mathcal{R}_j^{\frac{1}{2}}(\hat{p},x)\mathcal{C}_{j,k}(\hat{p},-x)\mathcal{R}_k^{\frac{1}{2}}(\hat{p},-x) \quad \text{real orthogonal matrix}$$

Solution of moment map constraint: $g_R = Q(\hat{p})\hat{T}$ with any $\hat{T}\in\mathbb{T}_n$, $|v_j|^2 = \frac{n\sinh\frac{x}{2}}{\sinh\frac{nx}{2}}\mathcal{R}_j(\hat{p},x)e^{-2\hat{p}_k}$. Spectral-gap condition from $|v_j|^2\geq 0$.

If we restrict to submanifold $\hat{p}_k-\hat{p}_{k+1}>\frac{|x|}{2}$, $\quad \forall k=1,\dots,n-1$, then requiring $v_j>0$ for all j gives complete gauge fixing. Writing $\hat{T}=e^{\mathrm{i}\hat{q}}$, the components of $\hat{p}$ and $\hat{q}$ provide Darboux coordinates on dense, open submanifod of the reduced phase space. Hamiltonian

$$\frac{1}{2}\mathrm{tr}\left(g_R(\hat{p},\hat{q})+g_R^{-1}(\hat{p},\hat{q})\right) = \sum_{k=1}^{n}(\cos\hat{q}_k)\prod_{m\neq k}\left[1-\frac{\sinh^2\frac{x}{2}}{\sinh^2(\hat{p}_k-\hat{p}_m)}\right]^{\frac{1}{2}}$$

is just $\widehat{H}_{\mathrm{RS}}(\hat{p},\hat{q})$.

Final remarks on the dual model

The ‘dual Lax matrix’ g_R generates commuting ‘dual flows’ globally.
On dense, open submanifold where $\hat{p}_k - \hat{p}_{k+1} > \frac{|x|}{2}$, can introduce

$$L^{\text{dual}}(\hat{p},\hat{q}) := \hat{\delta}(\hat{p},\hat{q})\, g_R(\hat{p},\hat{q})\, \hat{\delta}(\hat{p},\hat{q})^{-1}$$

with (non-unitary) diagonal matrix $\hat{\delta}_j(\hat{p},\hat{q}) := e^{\mathrm{i}\hat{q}_j/2}\mathcal{R}_j^{\frac{1}{4}}(\hat{p},-x)\mathcal{R}_j^{-\frac{1}{4}}(\hat{p},x)$

$$L^{\text{dual}}_{kl} = \frac{e^{\frac{\mathrm{i}\hat{q}_k+\mathrm{i}\hat{q}_l}{2}}\sinh\frac{x}{2}}{\sinh(\hat{p}_k-\hat{p}_l-\frac{x}{2})}\prod_{m\neq k}\left[1-\frac{\sinh^2\frac{x}{2}}{\sinh^2(\hat{p}_k-\hat{p}_m)}\right]^{\frac{1}{4}}\prod_{s\neq l}\left[1-\frac{\sinh^2\frac{x}{2}}{\sinh^2(\hat{p}_l-\hat{p}_s)}\right]^{\frac{1}{4}}$$

$L^{\text{dual}}(\hat{p},\hat{q})$ resembles the ‘original Lax matrix’ $\mathbf{L}(q,p)$ very much:

$$\mathbf{L}_{kl} = \frac{e^{\frac{p_k+p_l}{2}}\sinh\frac{x}{2}}{\sinh(\mathrm{i}q_k-\mathrm{i}q_l+\frac{x}{2})}\prod_{m\neq k}\left[1+\frac{\sinh^2\frac{x}{2}}{\sin^2(q_k-q_m)}\right]^{\frac{1}{4}}\prod_{s\neq l}\left[1+\frac{\sinh^2\frac{x}{2}}{\sin^2(q_l-q_s)}\right]^{\frac{1}{4}}$$

Hyperbolic (trigonometric) functions are replaced by trigonometric (hyperbolic) analogues. Domains of respective variables differ.

The dual flows are complete only on the full reduced phase space!
/We effortlessly obtain integration algorithm for the dual flows./

Concluding remarks

We described a mathematically motivated straight path leading to the Ruijsenaars-Schneider models: **follow natural generalizations from ordinary to Poisson-Lie symmetry and reduce canonical free systems.** This simplifies parts of the original works on these models, which used direct methods and were physically motivated by connections to solitons and relativistic invariance.

The results about the dual of the trigonometric RS model will be described in the forthcoming sequel to arXiv:0809.1509 [math-ph].

Some problems that we plan to investigate in the future:

- Find reduced systems at arbitrary moment value - spin RS models.
- Deal with hyperbolic and maybe also with elliptic RS models.
- Quantum Hamiltonian reduction ($\sim$ works on special functions).
- Derive $BC(n)$ models (and their Lax pairs) in analogous manner.