Equation of state for distributed mass quark matter

T S Biró, P Lévai, P Ván and J Zimányi

KFKI Research Institute for Particle and Nuclear Physics, H-1525 Budapest PO Box 49, Hungary

E-mail: tsbiro@sunserv.kfki.hu

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Abstract

We investigate how the QCD equation of state can be reconstructed by a continuous mass distribution of non-interacting ideal components. We find that adjusting the mass scale as a function of the temperature leads to results which conform to the quasi-particle model, but a temperature-independent distribution also may fit lattice simulation results. We interpret this as a support for the quark coalescence approach to quark matter hadronization.

(Some figures in this article are in colour only in the electronic version)

1. Distributed mass quark matter

The quark gluon plasma, produced in the Big Bang or in high energy relativistic heavy-ion collisions, hadronizes. We may detect and study this particular matter by observing signs of its collective behaviour, in local equilibrium obtaining its equation of state. Several statistical and hydrodynamical models have been considered in the soft QCD sector to describe hadron spectra.

We have developed a massive quark matter coalescence picture [2] which determines hadron yields and transverse spectra according to branching ratios between concurrent hadronization channels. Partonic level models of heavy-ion reactions also utilized the quark coalescence picture recently [3–5]. The seeming entropy reduction problem by coalescence with an associated reduction (confinement) of colour degrees of freedom can be resolved by assuming sufficiently massive partons around the hadronization temperature in the precursor matter. The necessary mass scale for quarks is about 300–350 MeV and even higher (about 700 MeV) for gluons. In [2] the explicit gluonic degrees of freedom were neglected.

In order to compose low hadron masses from massive constituent quarks we have introduced distributed mass partons into our hadronization model [6]. The low mass can be obtained from the convolution of distributed masses.

This approach led us to the investigation of an ideal but distributed mass parton gas. In particular we study (i) the equation of state (EoS) for a continuous mixture, (ii) the consistency of the quasi-particle picture, (iii) the fit of different mass spectra to lattice EoS data, and
(iv) we collect arguments in favour of a mass gap. We make a few interesting comparisons to non-ideal gas effects: a fixed mass $M$, scaling linearly with the temperature in the quasi-particle picture, is known to lead to a reduced pressure compared to the ideal gas (Stefan–Boltzmann) case even at infinite temperature $p/p_{SB} < 1$ [7–9]. This approach has been motivated by partially resummed high-temperature pQCD calculations. A pressure reduction even at high temperatures also occurs in non-perturbative lattice QCD calculations.

Slightly below $T_c$ the pressure does not vanish exactly. While earlier it was attributed to the finiteness of the modelled lattice, modern scaling techniques along the line of constant physics lead us to believe that the nonzero pressure here is a real effect. In fact massive resonance gas EoS fits quite nicely this emerging part in the temperature range $(0.8–1.1)T_c$ [9]. We re-evaluate these phenomena in the light of the distributed parton mass model.

The particle spectrum is given as a convolution integral of the spectral function $\rho(s)$ and a statistical factor $f$. The former may be characterized by dynamical factors, as e.g. a mass scale $M$, the latter by the temperature $T$ and chemical potential $\mu$ which are characteristic to the medium. An on-shell quasi-particle is described by a delta function with an arbitrary dispersion relation $E_p$: $\rho(E_p, \vec{p}) = \frac{1}{2E_p} \delta(E - E_p)$. The finite-width quasi-particles considered in [10] are more complex; they lead to a high cross section and viscosity to justify a strongly interacting liquid picture of quark matter near $T_c$. The distributed mass parton, considered here, is equivalent to a continuous spectral function: $\rho(s) = \int d m w(m) \delta(m^2 - s) = w(\sqrt{s})$, but this special ansatz does not consider the finite width of individual particles due to interaction in a given Lorentz frame. The continuous mass distribution accommodates modifications in the bulk pressure, entropy, etc. We note that field theoretical in-medium spectral functions break the Lorentz covariance and show a separate dependence on the energy $E$ and momentum $\vec{p}$.

2. Consistent equation of state with mass distribution

Thermodynamical consistency of the quasi-particle picture imposes further constraints on the mass distribution, $w(m)$. This can best be seen when starting with a homogeneous equation of state, given as a continuous sum of partial pressure contributions supported by a mean-field part:

$$p(T, \mu) = \int dm w(m) p_m(T, \mu) - \Phi(T, \mu).$$

(1)

Here the partial contributions are given by the ideal gas formula at a fixed mass,

$$p_m(T, \mu) = \int \frac{dE_k}{2\pi^2} \frac{k^3}{3} f \left( \frac{E_k - \mu}{T} \right)$$

(2)

with $f(x)$ being either the Fermi or the Bose (or approximately the Boltzmann–Gibbs) distribution and with a particular, $m$-dependent dispersion relation for a free particle $k = \sqrt{E_k^2 - m^2}$. The entropy density and number density, as respective derivatives of the pressure now contain extra terms due to the $T$- and $\mu$-dependence of the mass distribution and of $\Phi$, resulting in the following energy density, $e = Ts + \mu n - p$:

$$e = \int dm w(m) e_m(T, \mu) + \int dm T \frac{\partial w}{\partial T} p_m + \mu \frac{\partial w}{\partial \mu} p_m + \left( \Phi - T \frac{\partial}{\partial T} - \mu \frac{\partial}{\partial \mu} \right).$$

(3)

The quasi-particle consistency requires that the total energy is also a sum of the respective individual contributions plus the mean-field contribution, $\Phi$. Therefore the mass distribution $w(m)$ has to satisfy nontrivial constraints

$$\int dm \frac{\partial w}{\partial T} p_m(T, \mu) = \frac{\partial \Phi}{\partial T}, \quad \int dm \frac{\partial w}{\partial \mu} p_m(T, \mu) = \frac{\partial \Phi}{\partial \mu}.$$  

(4)
In order to obtain $\Phi(T, \mu)$ the integrability condition, $\frac{\partial^2}{\partial T \partial \mu} \Phi = \frac{\partial^2}{\partial \mu \partial T} \Phi$, has to be satisfied. This, assuming that $w(m; T, \mu)$ is integrable, leads to

$$\int dm \frac{\partial w}{\partial T} n_m(T, \mu) = \int dm \frac{\partial w}{\partial \mu} n_m(T, \mu).$$

(5)

The trivial way to satisfy this is to use a $T$- and $\mu$-independent mass distribution, $w(m)$. In this case the mean-field part, $\Phi$, is also $T$- and $\mu$-independent and reduces to an old-fashioned bag constant. The next step is the reconstruction of the equation of state. The pressure (and energy density) modification $\Phi(T, \mu)$ is obtained from integrating the constraint equations (4). The total pressure (equation (1)) becomes

$$p(T, \mu) = \int_{T_c}^{T} dT \int dm w(m, T, \mu) s_m(T, \mu) - \Phi(T_c, \mu).$$

(6)

The mean-field term, $\Phi$, cancels in the combination of $e + p$. This fact can help one to guess an appropriate mass distribution consistent with lattice EoS data, as well as with a quasi-particle picture.

We consider from now on a particular class of mass distributions, which depend on the thermodynamical environment parameters $T$ and $\mu$ only through a single mass scale, $M(T, \mu)$: $w(m, T, \mu) = \frac{1}{M^f(m/M)}$. The normalization integral for $w$ is inherited by the shape (form factor) function $f(t)$: $\int w(n) dm = \int f(t) dt = 1$. All medium dependencies are concentrated on $M(T, \mu)$, which should satisfy the constraint stemming from the integrability condition equation (5).

Now we try to guess the proper mass distribution in order to arrive at a pressure resembling lattice QCD simulation results. A particular, analytically integrable ansatz for the mass distribution shape is given by

$$f(t) = A \exp(-a t - b/t).$$

(7)

This ansatz is designed to suppress both low and high masses, while having a finite distributional width. The physical motivation behind suppressing low mass particles will be discussed in chapter 4. The normalization can be obtained from the general formula

$$\int_{0}^{\infty} dt t^{p-1} \exp(-a t - b/t) = 2(b/a)^{p/2} K_p(2\sqrt{ab}).$$

(8)

The pressure contribution normalized by the Stefan–Boltzmann pressure, $p_{SB} = p$, is given by

$$\sigma(g) = \int_{0}^{\infty} w(m) \frac{p_m}{p_{SB}} dm = \int_{0}^{\infty} dt f(t) \frac{g^2 t^2}{2} K_2(gt),$$

(9)

with $g = M/T$, is given by

$$\sigma(g) = g^2 \frac{\partial^3}{\partial a^3} \left[ K_2(\sqrt{b}(\sqrt{a + g} + \sqrt{a - g})) K_2(\sqrt{b}(\sqrt{a + g} - \sqrt{a - g})) \right].$$

(10)

In figure 1 we show the entropy density normalized by the Stephan–Boltzmann value (belonging to the massless ideal gas), $s/s_{SB} = (e + p)/(e + p)_{SB}$ as a function of the temperature (left) and the corresponding mass distributions (right).

3. Fits to lattice QCD results

The lattice data of $p/p_{SB}$ can be related to $\sigma(g)$ by fitting the results, with the assumption of a temperature-independent mass scale $M = T_c$, so that $g = T_c / T$. Since due to equation (9) $\sigma(0) = \int_{0}^{\infty} f(t) dt$, the normalization of the mass distribution requires $\sigma(0) = 1$. In this case $p = p_{SB}$ would be satisfied at infinite temperature. As a matter of fact lattice data
reach only $p/p_{SB} = 0.8$ at $T \approx 4T_c$ instead. The quasi-particle model with a fixed mass which is growing proportional to the temperature actually predicts such a deviation even in the infinite-temperature limit: in this case $\sigma(0) < 1$. Therefore it is important to judge without prejudice whether lattice QCD data suggest a $\sigma(0)$ value lower than one or not.

In our understanding a plot in terms of $g = T_c/T$ mediates a better picture of the extrapolation to the infinite temperature point at $g = 0$. The lattice QCD EoS data do not contradict to $\sigma(0) = 1$. It is of course still imaginable that a bending down occurs at high values of $T$ (low values of $g$) not simulated so far. The tendency with a growing lattice size (i.e. the comparison of $N_t = 4$ and $N_t = 6$ data), however, seems to support $\sigma(0) = 1$, a tautological consequence of the distributed mass model. While this high-temperature behaviour is well fitted by the pure exponential $\sigma(g)$ function, the part below $T_c$ cannot be recovered this way. A more steeply rising trial function is needed. Figure 2 plots the expectation value of the mass as a function of the temperature for the mass distribution (7) obtained by adjusting the pressure to the lattice result. Figure 3 presents the fitted pressure to lattice data and the interaction measure, $(e - 3p)/p_{SB}$. The mass distribution (7) as a function of mass $m$ and temperature $T$ is plotted in figure 4.
At the end of this section we consider the mass distribution (7) and investigate the temperature and baryochemical potential dependence of its mean mass scale parameter, $M(T,\mu)$.

The explicit showing of the other parameter(s), i.e. $b$, we shall suppress in the following: by plotting the expectation value of the mass instead of $M$ this is well founded. The entropy density becomes in this special case

$$s = \frac{\partial p}{\partial T} = \frac{A}{\pi^2} T^3 \left( 4\sigma \gamma - \frac{\mu}{T} \sigma \gamma' + \left( -\frac{M}{T^2} \sigma' + \frac{\partial M}{\partial T} \right) \gamma' \sigma' \right) - \frac{\partial \Phi}{\partial T}. \quad (11)$$

The quasi-particle consistency is equivalent to the principle that all terms, which were not there for a constant-mass, no-mean-field calculation, cancel:

$$\frac{A}{\pi^2} T^3 \frac{\partial M}{\partial T} \gamma' \sigma' = \frac{\partial \Phi}{\partial T}, \quad (12)$$
leaving us with
\[ s = \frac{A}{\pi^2} \left( 4 \sigma' - \frac{\mu}{T} \sigma' - g \gamma' \right). \]
Similarly the number density (here a conserved number density, in QGP the one third of the baryon charge density, i.e. the quark minus antiquark density) becomes
\[
n = \frac{\partial p}{\partial \mu} = \frac{A}{\pi^2} T^3 \left( \sigma' + \frac{\partial M}{\partial \mu} \gamma' \right) - \frac{\partial \Phi}{\partial \mu}.
\] (13)

The second quasi-particle consistency equation reads as
\[
\frac{A}{\pi^2} T^3 \frac{\partial M}{\partial \mu} \gamma' = \frac{\partial \Phi}{\partial T},
\] (14)
leaving us with
\[ n = \frac{A}{\pi^2} T^3 \sigma'. \] The energy density is combined to be
\[
e = T s + \mu n - p = \frac{A}{\pi^2} T^4 (3 \sigma - g \sigma') \gamma + \Phi.
\] (15)

It is noteworthy (although known for long) that the combination \(e + p\) is independent of the mean-field part:
\[
e + p = \frac{A}{\pi^2} \gamma (4 \sigma - g \sigma').
\] (16)

Now we investigate the integrability condition in this special case. The \(\mu\)-derivative of the \(T\)-consistency equation (12) (we shall call left-hand side, LHS) has to be equal to the \(T\)-derivative of the \(\mu\)-consistency equation (14). Omitting two common terms in both, namely \(\gamma' \partial^2 M/\partial T \partial \mu\) and \(\partial^2 M/\partial \mu \partial T \partial M/\partial \mu \gamma' \partial M/\partial \mu\), we are left with
\[
\text{LHS} = \frac{\partial}{\partial \mu} \frac{\partial \Phi}{\partial T} = \frac{A}{\pi^2} T^2 \gamma' \frac{\partial M}{\partial \mu},
\]
\[
\text{RHS} = \frac{\partial}{\partial T} \frac{\partial \Phi}{\partial \mu} = \frac{A}{\pi^2} T^2 \left( 3 \gamma \sigma' - \frac{\mu}{T} \sigma' - g \sigma' \right) \frac{\partial M}{\partial \mu}.
\] (17)

from the equality of the above expressions it follows
\[
\sigma' \gamma' \frac{\partial M}{\partial \mu} + \left( g \gamma \sigma'' + \frac{\mu}{T} \gamma' \sigma' - 3 \gamma \sigma' \right) \frac{\partial M}{\partial \mu} = 0.
\] (18)

In this partial differential equation for \(M(\mu, T) \sigma\) depends only on \(g = M/T\) and \(\gamma\) depends only on \(\alpha = \mu/T\). The solution can be characterized by \(M(T, \mu) = \text{constant lines on the } T - \mu \text{ plane. Such curves follow an ordinary differential equation obtained from (18)}:
\[
\frac{d \mu}{dT} = \frac{\gamma}{\gamma'} \left( g \frac{\sigma''}{\sigma} - 3 \right) + \frac{\mu}{T}.
\] (19)

The solution has the form \(\mu = T \alpha(g)\) whence we obtain
\[
\frac{1}{\gamma} \frac{d \gamma}{d \alpha} \left( \frac{d \mu}{dT} - \frac{\mu}{T} \right) = g \frac{\sigma''}{\sigma} - 3.
\] (20)

The expression in the bracket above reduces to
\[
\frac{d \mu}{dT} - \frac{\mu}{T} = \frac{\partial}{\partial T} \frac{\partial \alpha}{\partial g} = -g \frac{d \alpha}{dg}
\] (21)

while \(g T = M\) is constant. Therefore we obtain the following differential equation:
\[
- \frac{g}{\gamma} \frac{d \gamma}{d \alpha} \frac{d \alpha}{dg} = g \frac{\sigma''}{\sigma} - 3,
\] (22)

which becomes an integrable problem for \(\gamma(g)\):
\[
- \frac{d}{dg} \ln \gamma = \frac{\sigma''}{\sigma} - 3.
\] (23)
Its solution is given by
\[ \gamma = K g^3 / \sigma'(g). \]  

This, replacing the definition of \( \gamma \) in terms of \( \mu/T \), and \( g = M/T \) with constant \( M = m_0 \) one obtains a definite integral curve starting at \( T = T_0 \) for \( \mu = 0 \). That means that the constant \( K \) is related to \( T_0 \). By inverting the \( \cosh \) function the following explicit solution emerges for the \( M = m_0 \) constant lines:
\[ \gamma = T \ln \left( z(T) + \sqrt{z^2(T) - 1} \right) \]  

with
\[ z(T) = \left( 1 + \frac{\gamma_G}{2\gamma_Q} \right) \frac{T_0^3 \sigma'(m_0/T)}{T^3 \sigma'(m_0/T)} - \frac{\gamma_G}{2\gamma_Q}. \]  

The temperature dependence of the mass scale at \( \mu = 0 \), \( M(T, \mu) = m_0 \), has to be inverted in order to obtain \( T_0(m_0) \). This positions the crossings of \( M = m_0 \) constant characteristics with the temperature axis. We note that in every model with baryon–antibaryon symmetry these lines have zero \( \mu \)-derivatives at \( \mu = 0 \). The \( M(T, \mu) = m_0 \) characteristic lines in the temperature–chemical potential plane are plotted in figure 5.

4. Arguments for a mass gap

The \( \sigma (g) \) function in the Boltzmann approximation is given by (9). This is a so-called Meijer \( K \)-transform (a generalized Laplace transform) which can be inverted by
\[ f(t) = \frac{2}{i\pi} \int_{c-i\infty}^{c+i\infty} \sigma (g) \frac{I_2(gt)}{gt} \, dg. \]  

This presents a peculiar problem, whether there exists a unique mass distribution \( f(t) \), with the mass scale parameter kept temperature and chemical potential independent, to any \( \sigma (g) \) function extracted from an equation of state (e.g., from lattice QCD calculations). The shape of such a mass distribution is not arbitrary. We shall explore this possibility in a future work.

Utilizing the parametric integral (9) one easily derives a useful relation between moments of the mass distribution, \( f(t) \) and the EoS fit. We obtain
\[ M_n = \langle t^n \rangle_f = \frac{1}{\sqrt{\pi}} \frac{\Gamma \left( \frac{n+1}{2} \right)}{\Gamma \left( \frac{n}{2} + 2 \right) \Gamma (n)} (g^{n-1})_\sigma = c_n (g^{n-1})_\sigma. \]
A roughly approximate, qualitatively correct fit to the lattice EoS data is represented by the straight line, $\sigma(g) = 1 - g/g_c$, with $g_c \approx 1.15$. All the $n > 0$ moments of this expression are finite, meaning that the inverse mass moments are also finite:

$$\int_0^\infty f(t) \frac{dt}{t^n} = M_n = \frac{c_n g_c^n}{n(n + 1)}$$

This is possible only if the mass distribution has a finite mass gap, or it approaches zero more than any polynomial in the inverse mass $1/t$. Both possibilities represent an interesting spin-off of the EoS studies. The sizable reduction of the pressure at low temperature (large $g$), which causes the $n > 0$ moments of $\sigma(g)$ be finite in general, is related to confinement. This requires finite moments of the inverse mass with the mass distribution, which is possible only with a corresponding suppression of the low mass part, like $\exp(-M/m)$, or with a mass gap.

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