Statistical Power-Law Spectra due to Reservoir Fluctuations and the Universal Thermostat Independence Principle

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LHC ALICE data are interpreted in terms of statistical power-law tailed p_T spectra. As explanation we derive such statistical distributions for particular particle number fluctuation patterns in a finite heat bath exactly, and for general thermodynamical systems in the subleading canonical expansion approximately. Our general result $q = 1 - 1/C + \Delta T^2/T^2$ demonstrates how the heat capacity and the temperature fluctuation effects compete, and cancel in the standard Gaussian approximation. A new entropy formula, $K(S)$, is constructed for achieving $q_K = 1$. For huge fluctuations beyond the Gaussian ones, canonically the Gompertz cumulative distribution emerges.

I. INTRODUCTION

Power-law tailed distributions occur in Nature numerous. The idea of a statistical – thermodynamical origin of these emerged already decades ago [1, 2]. We have, however, long missed a "naturalness" argument connecting the basic principles of classical thermodynamics to the use of non-extensive entropy formulas by deriving canonical distributions of the one-particle energy. Although the observation has been made that the Tsallis and Rényi entropy formulas both lead to the cut power-law canonical distribution, and their use requires a constant heat capacity reservoir [3], the $q > 1$ power-laws – featuring a negative power of a quantity larger than one – still seem unnatural.

In our recent studies of ideal gases [4] we investigated energy fluctuations in a subsystem – reservoir couple. They lead to Tsallis distribution with $q = 1 - 1/C$ for ideal gas reservoirs, with C being the heat capacity of the total system.

Moreover, particle number fluctuations in the reservoir, either achieved naturally in a huge, inhomogenous heat bath or artificially by averaging the statistics over repeated events in high-energy experiments, lead to further effects [5–8]. We review in this paper how ideal fermionic and bosonic reservoirs, with binomial (BD) and negative binomial (NBD) distributions of the particle number, lead exactly to Tsallis power-law behavior with the parameters $T = E/\langle n \rangle$ and $q = \langle n(n-1) \rangle / \langle n \rangle^2$, when the microcanonical ideal gas statistical factor, $(1-\omega/E)^n$ in one dimension for massless partons, [34] is averaged over one of these distributions. The above q, named as second factorial moment, F_2, was determined with respect to canonical suppression in Refs. [9, 10]. For the binomial distribution one gets $q = 1 - 1/k$, for the negative binomial $q = 1 + 1/(k+1)$.

We demonstrate by fits to recent ALICE data taken in LHC experiment [11] that in the p_T-distribution of charged hadrons (dominated by pions) two Tsallis distributions emerge for the one-particle energy in a moving system, $\omega = \gamma(m_T - vp_T)$ (with $\gamma = 1/\sqrt{1-v^2}$ being the Lorentz factor and v a radial blast wave velocity, $m_T = \sqrt{m^2 + p_T^2} \approx |p_T|$ the so called transverse mass). The softer parts, below $p_T \approx 4$ GeV/c, show a dependence on the participant number as expected from statistical considerations: bigger systems come closer to the Boltzmann-Gibbs prediction.

Our theoretical results on q and T expressed by the mean multiplicity and its variance in the reservoir for BD and NBD distributions also can be viewed as an approximation for arbitrary particle number distributions in the reservoir up to subleading (second) order in the canonical expansion $\omega \ll E$. For non-ideal systems the general expansion up to second order delivers $q = 1 - 1/C + \Delta T^2/T^2$, a combined result with the heat capacity and the variance of the temperature of finite heat bath. These quantities seem to act against each other. Here the variance of the temperature is meant for the estimator $1/S'(E)$ of the thermodynamical temperature, the latter defined by $1/T = \langle S'(E) \rangle$. This way in the Gaussian approximation $\Delta T/T = 1/\sqrt{C}$ we regain $q = 1$ and verify the Boltzmann – Gibbs statistical factor. Part of this result has been derived and promoted by G. Wilk and Z. Wlodarczyk ($q = 1+\Delta T^2/T^2$) in recent years [12–14]. Instead of temperature fluctuations reservoir volume and particle number fluctuations were considered in recent publications [7, 8, 15, 16].

In order to generalize the canonical procedure we demonstrate that a deformed entropy $K(S)$ can be constructed and used for demanding $q_K = 1$ in the same approximation, practically using a canonical expansion with vanishing second order term. This requirement we call *Universal Thermostat Independence* Principle (UTI) [17]. The final entropy formula contains the Tsallis, Rényi and Boltzmann – Gibbs expressions as particular cases. Surprisingly there is another limit, that of huge reservoir fluctuations, $C\Delta T^2/T^2 \to \infty$, when the low-probability tails, canonical to this entropy formula, approach the cumulative Gompertz distribution, $\exp(1 - e^x)$, also met in extreme value statistics [18–20].

II. p_T SPECTRA AT THE LHC

In high-energy physics the power-law tail in p_T spectra is traditionally fitted by cut power-laws, $(1 + ap_T)^{-b}$, conjectured to stem from the behavior of hadronization matrix elements. As a matter of fact, a statistical model also can be applied to the fragmentation functions which describe the yield of hadrons stemming from high-energy particle jets [21, 22]. The real unknown is the soft part, with low p_T momenta; here

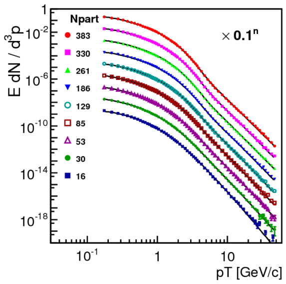

Figure 1: $(1 + ap_t)^{-b}$ fits to ALICE data on charged hadron p_T spectra in PbPb collisions[11] at LHC show two power-laws. Fit parameters as function of $N_{\rm part}$ are shown in Fig. 2.

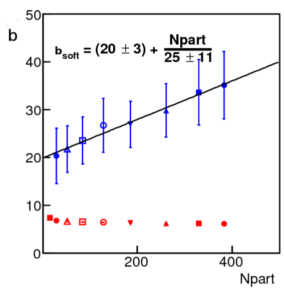

Figure 2: Powers in the power law, $b = 1/(q-1)$, follow a statistical trend for the soft spectra (upper symbols), while remain nearly constant for the hard spectra (lower symbols). The results belong to the participant numbers, $N_{\rm part}$, seen in the legend of Fig. 1.

thermal models are more fashionable.

It is therefore an intriguing question to decide whether there is a soft power-law, which can be naturally described and understood only by statistical phase-space considerations. The idea of a cut power-law as a thermal distribution, a characteristic consequence also from non-extensive thermodynamics, has been pursued by us since several years [23–25]. It is now for the first time that particle spectra over a wide p_T range are presented differentially for centrality classes [11]; such a presentation may inform about the multiplicity dependence of a heat reservoir in terms of thermal models.

In Fig. 1 we display our fits to p_T spectra of charged hadrons in centrality classes. A break in the spectra is pronounced at high centralities (large participant numbers, $N_{\rm part}$), which must be positively correlated with the particle number in the fireball where the hadrons were born. Our fits have the lowest χ^2 by making the soft – hard change around $p_T \approx 4$ GeV/c for all centrality classes, therefore we think it is justified to talk about soft and hard power-laws separately.

The fit parameter b, connected to the parameter q in Tsallis distribution as $b = 1/(q-1)$, is plotted against $N_{\rm part}$ in Fig. 2. The soft part shows a clear rising of the power b with $N_{\rm part}$, very characteristic to a statistical – thermal origin of a power-law. Contrary to this is the behavior of the hard spectra: the fitted power stays constant irrespective to the centrality, conjectured to vary with the size of the thermal bath. This is 'naturally' expected from QCD.

III. TEMPERATURE AND ENERGY FLUCTUATIONS

In this Section we turn to the theory of statistical power-law tailed distributions as canonical distributions in a thermal system connected to a heat reservoir with finite heat capacity. By fluctuation of temperature we mean the fluctuation of the estimator $1/S'(E)$ due to fluctuations of the energy E in the reservoir. We are interested in the observable distribution of the one-particle energy, $\omega \ll E$, in the canonical limit.

Traditionally such thermodynamical fluctuations are treated in the Gaussian approximation. Based on the fundamental thermodynamic uncertainty relation, $\Delta E \cdot \Delta\beta = 1$ with $\beta = S'(E)$, it is easy to derive the characteristic scaled fluctuation of the temperature [26–28]. With any well peaked distribution of a random variable, x, the expectation value $a = \langle x \rangle$ is near to the value where the peak occurs. As a consequence the variance of any function, $f(x)$ in this approximation is related to the original variance by a Jacobi determinant: $\Delta f = |f'(a)|\Delta x$. Now we consider both E and β as functions of the temperature, T. We obtain $\Delta E = |C|\Delta T$ with $C = {\rm d}E/{\rm d}T$ being the definition of heat capacity, and $\Delta\beta = \Delta T/T^2$. Combining these two results one arrives at the classical formula $\Delta T/T = 1/\sqrt{|C|}$. The heat capacity C is proportional to the heat bath size (volume, number of degrees of freedom) for large extensive systems.

There are, however, some deficiences in the Gaussian approximation. A Gauss distribution of β, given as $w(\beta) \propto \exp\left(-C(T\beta-1)^2/2\right)$, allows for a finite probability for negative temperatures, and – even worse – its characteristic function, $\left\langle e^{-\beta\omega}\right\rangle = \exp\left(-\omega/T + \omega^2/2CT^2\right)$ is not integrable in ω.

The next theoretical question is how to improve the canonical scheme beyond the Gauss approximation. We start our discussion with ideal gases. The one-particle energy, ω, out of total energy, E, is distributed according to a statistical weight factor $(1-\omega/E)^n$ [35]. The idea of superstatistics in general considers a distribution for the reservoir parameters n and E [29, 30]. In high-energy experiments E is typically controlled by the accelerator and does not vary much. However, n, the number of particles in the produced fireball scatters appreciably, which can be uncovered via the event-by-event detection of the spectra in ω, as suggested in [31].

In ideal reservoirs n particles are distributed among k phase-space cells: bosons $\binom{n+k}{n}$, fermions $\binom{k}{n}$ ways. The binomial and negative binomial distributions can be derived by considering a subspace (n,k) out of (N,K) in the limit $K \to \infty$ and $N \to \infty$ while $f = N/K$ is fixed.

$$F_{n,k}(f) := \lim_{K\to\infty} \frac{\binom{k}{n}\binom{K-k}{N-n}}{\binom{K}{N}} = \binom{k}{n} f^n (1-f)^{k-n}. \quad (1)$$

$$B_{n,k}(f) := \lim_{K\to\infty} \frac{\binom{n+k}{n}\binom{N-n+K-k}{N-n}}{\binom{N+K+1}{N}} = \binom{n+k}{n} f^n (1+f)^{-n-k-1}. \quad (2)$$

These distributions are normalized based on the binomial expansion of $(a+b)^k$ and $(b-a)^{-k-1}$, respectively.

Assuming a typical fireball in high-energy experiments, E is fixed and n fluctuates according to NBD. The ideal gas statistical weight factor, describing the complement phase-space for reservoir configurations, becomes [36]

$$\sum_{n=0}^{\infty}\left(1-\frac{\omega}{E}\right)^n B_{n,k}(f)=\left(1+f\frac{\omega}{E}\right)^{-k-1}. \tag{3}$$

Note that $\langle n\rangle=(k+1)f$ for NBD. Then with $T=E/\langle n\rangle$ and $q=1+1/(k+1)$ we get

$$\left(1+(q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}}. \tag{4}$$

This is *exactly* a $q>1$ Tsallis – Pareto distribution. Similarly in a fermionic reservoir E is fixed, n is distributed according to BD. We obtain

$$\sum_{n=0}^{\infty}\left(1-\frac{\omega}{E}\right)^n F_{n,k}(f)=\left(1-f\frac{\omega}{E}\right)^{k}. \tag{5}$$

Note that $\langle n\rangle=kf$ for BD. Then with $T=E/\langle n\rangle$ and $q=1-1/k$ we again get a Tsallis-Pareto distribution, but now with $q<1$. In the $k\gg n$ limit (low occupancy in phase-space) the particle distribution in the reservoir becomes Poissonian in both cases. The result is exactly the Boltzmann-Gibbs weight factor with $T=E/\langle n\rangle$:

$$\sum_{n=0}^{\infty}\left(1-\frac{\omega}{E}\right)^n \frac{\langle n\rangle^n}{n!}e^{-\langle n\rangle}=e^{-\langle n\rangle\,\omega/E}. \tag{6}$$

We note that NBD distributions are observed experimentally, a nice analysis of heavy ion data are given by the PHENIX group [32]. In all the three above cases

$$T=\frac{E}{\langle n\rangle}\quad\text{and}\quad q=\frac{\langle n(n-1)\rangle}{\langle n\rangle^2}. \tag{7}$$

Now we turn to the ideal statistical weight factor with general finite reservoir fluctuations. In the canonical approach we expand for small $\omega\ll E$ and view the Tsallis-Pareto distribution as an approximation:

$$\left(1+(q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}}=1-\frac{\omega}{T}+q\frac{\omega^2}{2T^2}-\dots \tag{8}$$

on the one hand and

$$\left\langle\left(1-\frac{\omega}{E}\right)^n\right\rangle=1-\langle n\rangle\frac{\omega}{E}+\langle n(n-1)\rangle\frac{\omega^2}{2E^2}-\dots \tag{9}$$

on the other hand. To match up to subleading canonical order, it follows in general:

$$T=\frac{E}{\langle n\rangle}\quad\text{and}\quad q=\frac{\langle n(n-1)\rangle}{\langle n\rangle^2}. \tag{10}$$

Finally we consider a general system with general reservoir fluctuations. Expanding for small $\omega\ll E$

$$\begin{aligned}\left\langle e^{S(E-\omega)-S(E)}\right\rangle_{\omega\ll E}&=\left\langle e^{-\omega S'(E)+\omega^2S''(E)/2-\dots}\right\rangle\\ &=1-\omega\langle S'(E)\rangle+\frac{\omega^2}{2}\left\langle S'(E)^2+S''(E)\right\rangle-\dots\end{aligned} \tag{11}$$

Compare this with the expansion eq.(8) of the Tsallis distribution: In the view of the above we interpret the parameters as

$$\frac{1}{T}=\langle S'(E)\rangle,\qquad q=\frac{\left\langle S'(E)^2+S''(E)\right\rangle}{\langle S'(E)\rangle^2}. \tag{12}$$

Here $\langle S''(E)\rangle=-1/CT^2$ follows from the definition of the heat capacity of the reservoir, $1/C=\mathrm{d}T/\mathrm{d}E$. Summarizing these results we understand that the parameter q has opposite sign contributions from $\left\langle S'^2\right\rangle-\left\langle S'\right\rangle^2$ and from $\langle S''\rangle$. In general

$$q=1+\frac{\Delta T^2}{T^2}-\frac{1}{C}. \tag{13}$$

to subleading canonical order. With this formula $q>1$ and $q<1$ are both possible and for temperature fluctuations with Gaussian variance, $\Delta T/T=1/\sqrt{C}$, one has $q=1$.

IV. DEFORMED ENTROPY FORMULAS

Techniques to handle the $q=1$ case are known since long. For dealing with $q\neq 1$ systems the calculations as a rule are involved, but the introduction of a deformed entropy, $K(S)$, instead of S provides more flexibility for handling the subleading term in ω [17, 33]. The deformed statistical weight has an average over the reservoir fluctuations, as follows

$$\begin{aligned}\left\langle e^{K(S(E-\omega))-K(S(E))}\right\rangle&=1-\omega\frac{\mathrm{d}}{\mathrm{d}E}K(S(E))\\ &+\frac{\omega^2}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}E^2}K(S(E))+\left(\frac{\mathrm{d}}{\mathrm{d}E}K(S(E))\right)^2\right).\end{aligned} \tag{14}$$

Note that $\frac{\mathrm{d}}{\mathrm{d}E}K(S(E))=K'S'$ and $\frac{\mathrm{d}^2}{\mathrm{d}E^2}K(S(E))=K''S'^2+K'S''$. Comparing this expansion with the Tsallis distribution we obtain the parameters for the deformed entropy. Using previous notations for averages over reservoir fluctuations but assuming that $K(S)$ is independent of these we obtain

$$\begin{aligned}\frac{1}{T_K}&=K'\frac{1}{T},\\ \frac{q_K}{T_K^2}&=\left(K''+K'^2\right)\frac{1}{T^2}\left(1+\frac{\Delta T^2}{T^2}\right)-K'\frac{1}{CT^2}.\end{aligned} \tag{15}$$

By choosing a particular $K(S)$ one manipulates q_k. After a simple division we obtain

$$q_K = \left(1 + \frac{\Delta T^2}{T^2}\right)\left(1 + \frac{K''}{K'^2}\right) - \frac{1}{C}\frac{1}{K'} \tag{16}$$

Not considering superstatistical (event by event) reservoir fluctuations one assumes $\Delta T/T = 0$. With such assumptions from $q_K = 1$ we arrive at the original UTI equation [17]:

$$\frac{K''}{K'} = \frac{1}{C}. \tag{17}$$

The solution of eq. (17) delivers $K(S) = C\left(e^{S/C} - 1\right)$ and one obtains upon using $K(S) = \sum_i p_i K(-\ln p_i)$ the statistical entropy formulas of Tsallis and Rényi:

$$K(S) = \frac{1}{1-q}\sum_i \left(p_i^q - p_i\right) \quad \text{and} \quad S = \frac{1}{1-q}\ln\sum_i p_i^q. \tag{18}$$

Finally we obtain a novel, general deformed entropy formula including superstatistical reservoir fluctuations. Demanding $q_K = 1$ one obtains the differential equation

$$C\,\frac{\Delta T^2}{T^2}K'^2 - K' + C\,\left(1 + \frac{\Delta T^2}{T^2}\right)K'' = 0. \tag{19}$$

The solution of eq. (19) is given by

$$K(S) = \frac{C_\Delta}{\lambda}\,\ln\left(1 - \lambda + \lambda e^{S/C_\Delta}\right). \tag{20}$$

with $\lambda := C\Delta T^2/T^2$ and $C_\Delta = C + \lambda$. Based on this the following generalized entropy formula arises:

$$K(S) = \frac{C_\Delta}{\lambda}\,\sum_i p_i\,\ln\left(1 - \lambda + \lambda p_i^{-1/C_\Delta}\right). \tag{21}$$

For $\lambda = 1$ (Gaussian temperature fluctuations) this expression leads exactly to the Boltzmann entropy, irrespective of the value of C_Δ. The same limit is achieved for infinite reservoirs, $C \to \infty$ while keeping λ finite; the entropy formula is traditional. For $\lambda = 0$ (no fluctuations in the reservoir) it becomes the Tsallis entropy with $q = 1 - 1/C$. Finally for $\lambda \to \infty$ (huge fluctuations) it transforms to the parameter free formula,

$$K(S) = \sum_i p_i \ln\left(1 - \ln p_i\right). \tag{22}$$

The canonical p_i distribution to this, is a Lambert W-function. It shows tails according to the cumulative Gompertz distribution, originally discovered in demographic models [18], and later used as a tumor growth model [19]. This distribution also occurs in studies of extreme value distributions, showing deviations from scaling in occurence frequencies of big earthquakes[20] .

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[34] This is $\exp(S(E-\omega) - S(E))$, the complement phase-space factor for ideal gases. Since each exponential grows like x^n, their ratio delivers the formula.
[35] This statistical weight factor is to be multiplied with a one-particle phase-space factor, $\rho(\omega)$, which however does not depend on n and E.
[36] One has to note that in fact zero particles in the reservoir are irrealistic. Partial sums starting not at $n = 0$ can be obtained by derivations with respect to ω.