A SIMPLE MATHEMATICAL MODEL FOR THE RISE AND FALL OF CIVILIZATIONS

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Abstract

The presented model of the development of civilizations, inspired by the best strategy game of all PC times by Sid Meier[1], is assumed to follow certain simple differential equations in the long term behavior. Starting with basic population dynamics we arrive at questions of peace and war, free trade, culture, science and technology in a deterministic single civ approach. A case study for parameter resetting reveals how to extract civilizing strategies from this study.

1 The mathematics of history

It is contemporarily still debated whether "*the*" history of civilized humanity can be modeled in mathematical terms at all, and if yes, whether we can select out exactly those good models, which allow for sufficiently rational conclusions overriding the spiritual and intellectual costs paid for constructing and exploring the very mathematical concepts necessary to make quantitative predictions on this subject. It seems to be worth to start this endeavor only then, if it is not a priori hopeless to conclude from simplified models, reduced to the quantitative description of a few characteristics of folks, communities and their civilizations, at a useful strategy for guiding our own civilization towards success. And to do this with a chance somewhat greater than just blind odds. Who has no such hope, should stop to read here.

Based on the above outlined attitude I list my basic assumptions about the history of human civilizations as follows.

- The history is one and unique. Its details, however, reveal statistical properties and there exist explorable rules, similar to other scientific problems like evolution of life, our planet Earth and cosmology.

[1] TM CIV

- The characteristic basic unit of the historical evolution process is the civilization. Civilizations have an autonomous, internal dynamics, established by natural laws of physics and biology and by available resources from the environment actually cultivated by that civilization, as well as external influences on this dynamics stemming form interactions with other civilizations.

- The most important fundamental order parameter for the autonomous evolution is the increase and decrease of the population. In the last ten thousand years of mankind this is the most prominent pointer. The basic equations of civil dynamics are equations of population dynamics[2], only their parameters are not exclusively of biological origin.

- The interactions among civilization units (empires, nations, city-states) modify the parameters of these basic equations, forming this way the most remarkable pointer, the population number. Increase of the rate of growth (acceleration) in general happens on the account of reducing the expansion rate of others (neighbors, enemies, friends). At the same time a rationalistic strategy of promoting the own growth has to be centered around a careful deliberation of possible gains and necessary losses.

- Random events, exceptional odds (wonders, great personalities, chance) may play an influential role, but already a deterministic model should be able to formulate predictions. These can be re-checked on past historical events, which were not part of fitting the model parameters. The uncertainty of this control, however, might be large, since past historians were not half as much interested in statistical aspects of history as in the entertaining value of their presentation. Our insights in the simple model presented below will be tested in CIV games with simple strategies.

- The human history up to now and presumably in the future shows a vague *epochal* structure. The reason for this feature may be some "bottle-neck" effect in a narrow range of optimal ensembles of development parameters.

In the CIV game the basic unit is a city (village, town, metropolis) with some (cultivated) area of influence and resource gain. The growth of population, in case of sufficient resources, is almost automatic, in case of insufficiency stagnation or famine may occur stopping or even reversing the population growth. The consumption is proportional to the number of people[3]. The area integrated production rate can be improved: either by enclosing new areas and with that new resources, or by erecting proper communal institutions, or by developing new technologies, or by increasing the ratio of specialized experts to ordinary workers and tax payers. Alternatively, in some state forms on the short run the rate can be increased also by spending state money or sacrificing a part of the population, but this money has to be earned first from taxes paid by

[2] Thomas Malthus has formulated this most cleanly.
[3] It is indicated in the game for each city on a logarithmic scale.

the sometimes unhappy population or from profit branched off from trade. A vigorous spiritual life and blooming culture has a beneficial influence on the common mood and by that on the stability of government and on the development projects mentioned above. All this activities are entangled. There is a room for rivalry as well as for cooperation with other civilizations in terms of wars, trade or diplomacy. As a thumb rule a flexible, intelligent reign, preemptive to ever changing challenges leads to victory (or at least helps to avoid extinction).

Even the CIV simulation itself is too complicated for a direct mathematical analysis (not mentioning real human history). Furthermore the equations of the simulation are not made public[4]. Here we aim to discuss something much simpler, a mean field model of a single civilization faced to an average rival environment of other civilizations. We hope that simplicity helps to learn something about this complex and fascinating subject.

2 The base: population dynamics

The starting point of my model investigates the dynamics of population. A population may grow or decrease due to a number of elementary biological reasons. The rate of growth is determined by the actual number of population and some constants parameterizing the quality of the life niche. The simplest general formula describing such a reproduction rate is given by

$$p(t+d) \;=\; p(t) \;+\; d\,(s-\mu)\,p(t) \tag{1}$$

with $p(t)$ being the population at time t, $t+d$ the next instant investigated (e.g. one generation - . 25 years - later), s the reproduction rate and finally μ the death rate (exit rate from the fertile population). Investigating time scales on which d is a small number the basic equation becomes a differential equation:

$$\dot{p} \;=\; (s-\mu)\,p. \tag{2}$$

According to the balance between the rates of reproduction and decease (more precisely the exit form the fertile population) the total number of peoples grows for $s > \mu$ and goes back for $s < \mu$. The reproduction dynamics is exponential, as it has been formulated by Malthus:

$$p_{Malthus}(t) = p_0 \exp\left((s-\mu)t\right). \tag{3}$$

Assuming a polygamic reproduction process, when every prospective partner is included in sexual activity, the growth rate of the population is even more

[4] An exception is freeciv by the Linux community. But it is a highly nontrivial task to reconstruct mathematical equations from the public source code.

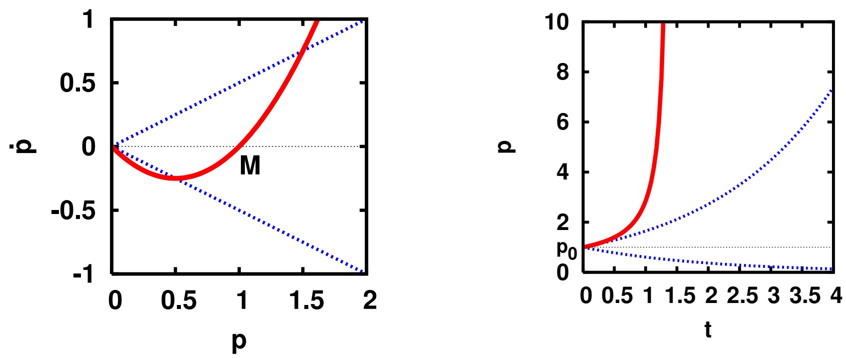

Figure 1: Population growth dynamics in monogamic and polygamic populations. The rate of increase $\dot{p}$ as a function of the number of individuals p reveals attracting or repelling fixed points. The time evolution curves start normalized, from $p = 1$. The parameters are $s = 1, \mu = 0.5$ for the exponential increase, $s = 1, \mu = -0.5$ for the exponential decrease and finally $\lambda = 1, \mu = 0.5$ for the divergent scenario (red line).

dramatic[5]. The reproduction rate s itself is proportional to the population $s = \lambda p$. In this case more and more people helps a wildly increasing number of heirs to existence with more and more partners. The heirs in a certain but high enough percentage are also fertile. There is a serious chance to perform a really uncontrolled growth of such a population. Without a loss rate ($\mu = 0$) the population number will diverge within a finite time:

$$\dot{p} = \lambda p^2 \tag{4}$$

has the solution

$$p(t) = \frac{p_0}{1 - p_0 \lambda t}, \tag{5}$$

which becomes infinity at the time instant $t = 1/p_0\lambda$ for any starting value p_0. This model must have some shortcomings. The little more sophisticated equation, taking into account a finite death rate, is

$$\dot{p} = \lambda p^2 - \mu p. \tag{6}$$

This means growth if $p > \mu/\lambda$, otherwise stagnation or decrease. The population must have achieved a minimal number in order to avoid extinction. Beyond this Malthus point starts the unlimited growth. Such cases must have ample precedence in biological systems.

[5]Probably it is not accidental, that this scenario was not considered by the Victorian Malthus.

Even with finite μ the population diverges after a finite time:

$$p(t) = \frac{p_0 p_M e^{-\mu t}}{(p_M - p_0) + p_0 e^{-\mu t}} \tag{7}$$

for $p_0 > p_M = \mu/\lambda$ formally leads to $p(t_{\text{div}}) = \infty$ after passing the time

$$t_{\text{div}} = \frac{1}{\mu} \ln \frac{p_0}{p_0 - p_M}. \tag{8}$$

3 Limits of growth: consume

The reproduction behavior of real human populations is neither totally monogamic nor hundred percent polygamic. As it will be shown at the end of this paper, already a weak degree of polygamy leads to the same qualitative behavior: below a certain minimal number the population dies out. In this section we shall, however, experience that taking the limits of growth[6] into account there is no qualitative difference between monogamic and polygamic societies in the population saturation behavior.

The growth is stopped by the exhausting of resources, which is followed by a decrease in the reproduction ability. Both the monogamic rate s and the polygamic rate λ is proportional to the area A from which resources (raw material, food, energy) are gained and to their occurrence density r (the latter including as a factor the level of technology practiced by the civilization). This is important, this makes the rivalry for controlling areas to a fundamental factor in civilization dynamics. The reproduction ability is decreased due to consumption, due to use of the resources for purposes other than biological reproduction. The c proportionality constant in this term is the 'per capita' luxury waste. The so far constant parameters become linear functions of the population in this next level approximation:

$$\begin{aligned} \dot{p} &= (rA - cp)\, p - \mu p, \\ \dot{p} &= (rA - cp)\, p^2 - \mu p. \end{aligned} \tag{9}$$

In both cases a saturation number, p_S, occurs signaling the maximal use of resources. Below this point the population increases above it decreases, so this is a stable fixed point (cf. Figure 2.). After long enough time the population number saturates to this S-point if the resource usage rA surpasses a minimal value, in the opposite case the civilization is unsuccessful and the population dies out.

The saturation value of the population becomes

$$p_S = \frac{rA - \mu}{c}, \tag{10}$$

[6]Club of Rome

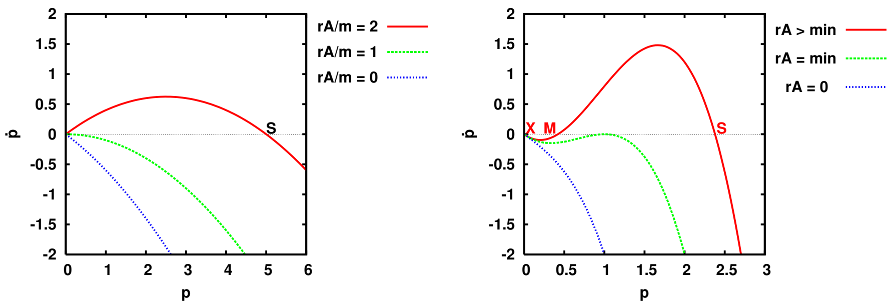

Figure 2: Population dynamics with reproduction rates depending on the population linearly. The growth rate $\dot{p}$ as a function of the number p reveals attracting (stable) and repelling (unstable) fixed points. On the red line to the right X denotes the extinction point, M the Malthus point, beyond that the population growth sets in, and finally S the saturation point, where an equilibrium between reproduction and consumption is established. The condition for not to become extinct are $A > \mu/r$ in the monogamic and $A > 2\sqrt{\mu c}/r$ in the polygamic demographic scenario, respectively.

in the monogamic case, and

$$p_S = \frac{rA}{2c}\left(1 + \sqrt{1 - \frac{4\mu c}{r^2 A^2}}\right). \tag{11}$$

in the polygamic case. In the first case saturation is possible if $A > \mu/r$, i.e. the resources collected from the total controlled area (rA) overcome the effect of natural death (aging, emigration, etc.). The second case is even more interesting; now the minimal resource value has to exceed twice the geometrical mean of the per capita state consumption and mortality rate, $rA > 2\sqrt{\mu c}$, otherwise de-population follows[7]. When the area and the effectivity of gaining resources from it are large, then the saturation number can be approximated by $p_S \approx rA/c$, while the Malthus point shrinks towards a small number, $p_M \approx \mu/rA$. Between these two points a phase of growth occurs, the golden age of a civilization. It is obvious that for a given mortality rate μ and consumption c an elementary interest is to increase the product rA. The area, A can be increased by emigration followed by colonization or by conquest, the factor r can be boosted by technological development. The area also may be effectively increased without the physical motion of the own population by vigorous (free) trade[8]. These possibilities will be analyzed in later sections.

[7]E.g. overextended taxation may cause famine, emigration or increased crime.
[8]Free trade is also an act of aggressivity, but sometimes both parties enjoy it.

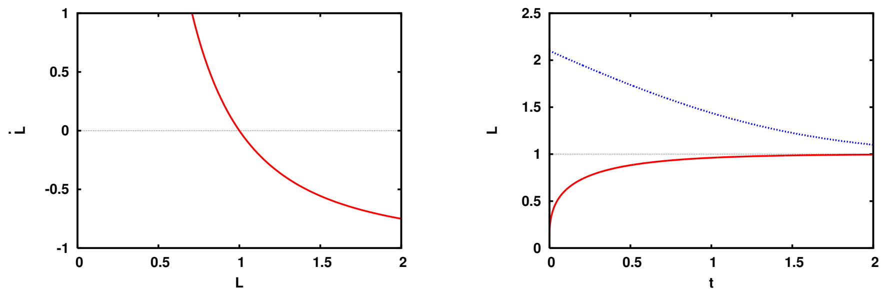

Figure 3: The dynamics of areal growth at fixed population. The attracting fixed point $L_S = \sqrt{m/p}$ is in this case at $L_S = 1$.

4 Areal dynamics: colonization

It is a simpler, faster process to increase the area under the control of a civilization than the effectivity of resource usage. It is more so at the beginning, at the dawn of civilization, when there is a lot of unused area (virgin land) but the technological evolution is slow. The factor A increases without any particular conflict with other civilizations simply due to emigration and settlement – shortly due to colonization.

We shall characterize the dynamical growth of the civilized area by the following equation:

$$\dot{A} = 2\kappa \left(\frac{p}{A} - m \right) \sqrt{A}. \tag{12}$$

In this model the local drive behind the colonization process is the population density p/A. If this exceeds a certain threshold value, m (stands for migration threshold), then the settled area grows. A further important feature is that the size of the frontier area helps the emigration: we assume that the rate of colonization is proportional to this measure. Since the area occupied by a civilization, i.e. the land, is two dimensional, the size of the frontier area is proportional to the square root of the area, $\sqrt{A}$. This is true for any convex shaped land[9].

For the sake of transparency it is worth to introduce the substitution $A = L^2$, and to regard the equation for the linear size L (limes):

$$\dot{L} = \kappa \left(\frac{p}{L^2} - m \right). \tag{13}$$

This equation (13) or equivalently the equation (12) couples to the demographic equation (9) in either the monogamic or the polygamic version.

One can assume with right that the area growth is faster than the change in the population (although by numerical computation the general case can be

[9]In general a fractal power of A smaller than 1/2 may also be considered

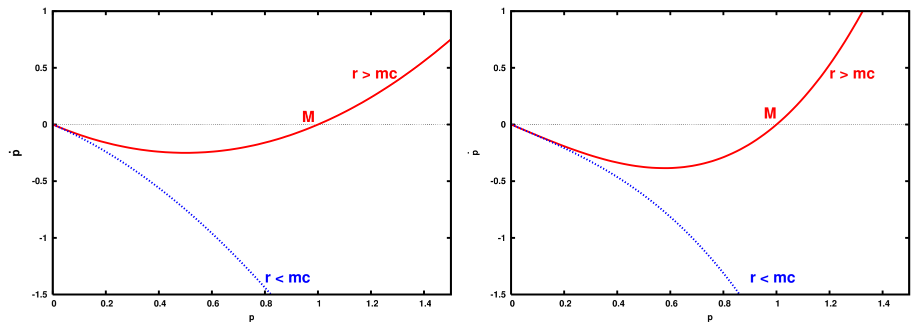

Figure 4: Population dynamics in the areal fixed point. The monogamic (left) and polygamic (right) reproduction strategy both leads to a Malthus point (M). The actual curves have been drawn for $r/mc = \pm 1$.

investigated, too). For the colonization a time shorter than a generation is sufficient. In this case the equation (13) can be analyzed at constant p.

Substituting the equilibrium area value $A_S = L_S^2 = m/p$ into the demographic equation (9) describing the slower process we arrive at an interesting result:

$$\begin{aligned} \dot{p} &= \left(\frac{r}{m} - c\right) p^2 - \mu p \\ \dot{p} &= \left(\frac{r}{m} - c\right) p^3 - \mu p. \end{aligned} \tag{14}$$

It is surprising but both the monogamic and polygamic practice leads to qualitatively similar demography (see Fig.4) in the rapid colonization scenario. Whenever a civilization possesses sufficient resources, i.e. $r > mc$, then a Malthus point occurs, below which extinction and beyond which an accelerating growth waits for the civilization. The growth is limited only by the finiteness of the total area for all civilizations introducing an era of wars among different cultures. If $r < mc$, so the consumption is too high relative to the resource usage efficiency, then the decline of the population is ensured. Alone the value of the Malthus point is different for the different demographics:

$$\begin{aligned} p_M^{\mathrm{MONO}} &= \frac{\mu}{\frac{r}{m} - c} \\ p_M^{\mathrm{POLI}} &= \sqrt{\frac{\mu}{\frac{r}{m} - c}}. \end{aligned} \tag{15}$$

4.1 The large area limit: empires

The common dynamics of the population p and the civilized area A can also be studied analytically in the limit of large area and fast colonization. Let us

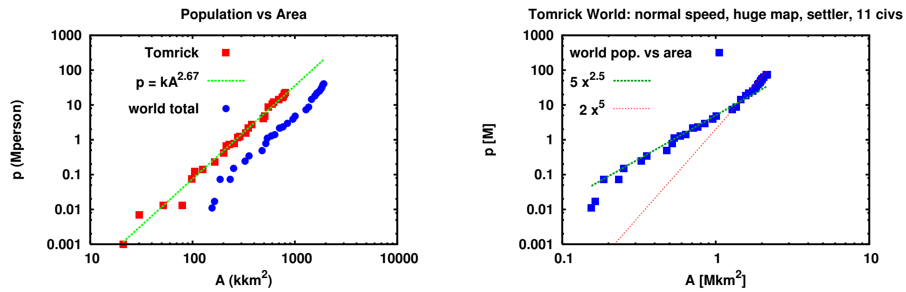

Figure 5: Population versus area in the gross phase of the civilization (Tomrick game: settler level, huge map, normal speed, no war). The selected civilization to the left, the world average of 11 civilizations to the right. The best fitted power of 2.67 belongs to $\alpha = 0.064$, to an almost monogamic demography.

consider the α-polygamic demographic equation

$$\dot{p} = (rA - cp)p^{1+\alpha} - \mu p, \tag{16}$$

where $\alpha = 0$ belongs to the monogamic and $\alpha = 1$ to the fully polygamic reproduction habit. We consider the fast growth limit; in this case the first term dominates $rA \gg cp$ and $rA \gg \mu$:

$$\dot{p} = rAp^{1+\alpha}. \tag{17}$$

The colonization is also assumed to be fast, the population density being well over the migration threshold $p/A \gg m$. In this case the area dynamics of eq.(12) can be approximated by

$$\dot{A} = 2\kappa p A^{-1/2}. \tag{18}$$

These equations allow us to consider the population as a function of the civilized area directly. We get

$$\frac{dp}{dA} = \frac{r}{2\kappa} A^{3/2} p^{\alpha}. \tag{19}$$

The analytic solution at asymptotically large area is given by

$$p \sim \frac{1}{1-\alpha} A^{\frac{5}{2}\frac{1}{1-\alpha}} \tag{20}$$

In particular in the monogamic scenario $\alpha = 0$ and one predicts a certain power law, $p \sim A^{5/2}$, in the early fast growing period of a civilization. For $\alpha = 1$ the solution is exponential, $p \sim \exp(A^{5/2})$ and for e.g. $\alpha = 1/2$ one obtains $p \sim A^5$.

We have tested this power law prediction in some CIV games. In Fig.5 the population p is plotted against the colonized area A in a double logarithmic scale. In this presentation the power law is represented by a straight line. The game data lie on the power-law line with the monogamic power 5/2 for a long period of evolution. After achieving the industrial age other synergic factors seem to be switched on, because the power increases to 5.

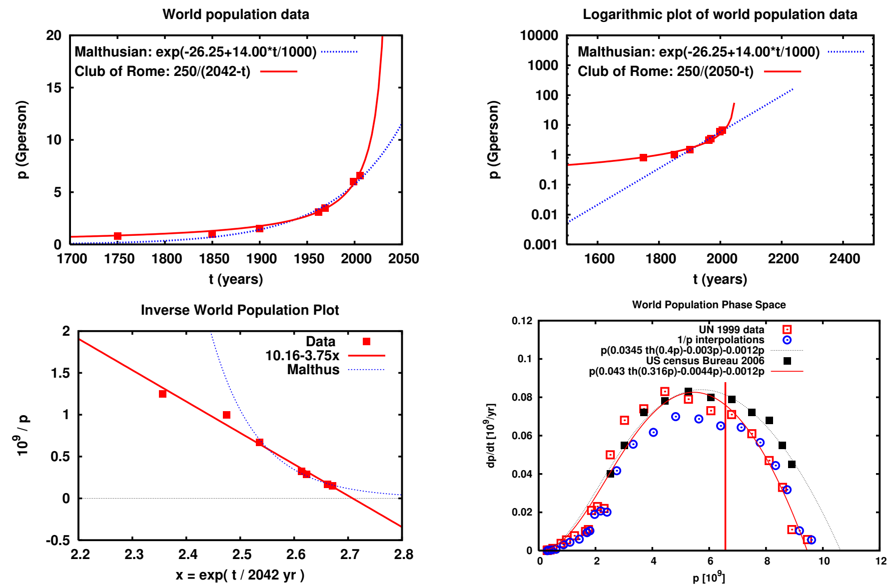

Figure 6: The increase of the world population based on UN and US Census Bureau data (which contain also predictions) and by our various assumptions. The different presentations range from the linear, logarithmic and inverse population time series to a phase space approach plotting the rate of change dp/dt against the actual population number p in the bottom roght corner. The thick red vertical line indicates the world population at the beginning of 2007.

4.2 Case study I: world population

We also tested the demographic equation on world population data, available on the internet from US Census Bureau and The United Nations Organization. The phase space presentation plotting $\dot{p}$ versus p allows for predictions in the most transparent way (cf. the bottom right plot on Fig.6). Although the spread of data is quite appreciable, a prediction of saturating Earth's population between 8 and 12 billion can be made. This prediction is based on the leveling off in the population growth rate $\dot{p}/p$ in the 1960-s. The different curves show the growth rate dp/dt versus the population p, counted the derivatives by several finite time difference formulas (linear in p, linear in $1/p$, etc.).

Another view at world population data is given in Fig.7

Since the mark $p_0 = 3$ billion has been achieved in 1960 a linear decrease trend is manifested in the annual growth percentage $\dot{p}/p = (rp_0/m - \mu) - cp$. This behavior belongs to a constant (most probably saturated) area of resource use by Earth's civilizations. The predictions by the UN follow this line, predicting a maximal population of about ten billion around 2100. In the earlier evolution the

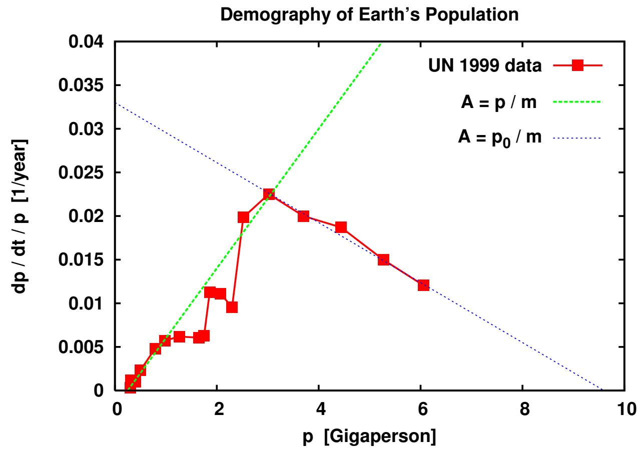

Figure 7: The world's demography in a logarithmic phase space showing $dp/dt/p$ against p. Since the mark 3 billion has been achieved a linear decrease trend is manifested in the annual growth percentage $\dot{p}/p$. This behavior belongs to a constant (most probably saturated) area of resource use by Earth's civilizations.

resource area growth seems to be well fitted by the fixed point of our colonization equation: $p = mA$. In this case $\dot{p}/p = (r/m - c)p - \mu$, for $r > mc$ giving a rising line. The dynamics $\dot{p} \sim p^2$ has lead the Club of Rome to the well-known conclusion that the population growth hyperbolically, diverging around 2050. Strange that this trend has just been inverted for the 1970-s.

The UN data from 1999 fit well these two lines. From this we obtain the following parameters for our demographic and area growth equations: The net population loss rate is about $\mu \approx 0.002$ yearly (0.2 per cent of the total population). The consumption factor is $c \approx 0.00344$ per year per billion person. The resource usage relative to the migration threshold becomes $r/m \approx 0.001144$ per year per one billion person. Finally the crossing of the lines occurs at $p_0 = 3.06$ billion. These numbers are, however, only approximate, as the difference between the red line and black line prediction indicates in the bottom right plot of Fig.6. These lines have been drawn by assuming a tangent hyperbolic smooth transition from the rising to the falling line in the $\dot{p}/p$ plot.

Estimating the civilization area as that of $A_0 = p_0/m \approx 10^8\ \mathrm{km}^2$ (i.e. 18% of the planets surface, about half of the land area), we obtain $m \approx 30.6\ \mathrm{person/km}^2$ as the migration threshold and $r \approx 0.35 \cdot 10^{-9}/(\mathrm{year\ km}^2)$ for the resource usage efficiency. The saturation of areal dynamics has occurred at $p_0 \approx 3$ billion, in the year 1960 according to the UN data of 1999.

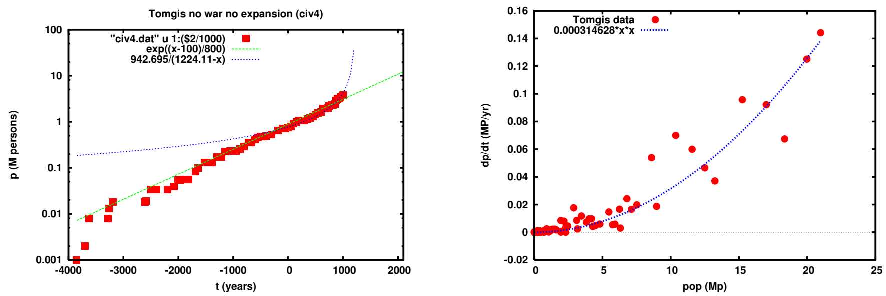

Figure 8: Population dynamics in the CIV IV game Tomgis (Settler level, mongol empire, no wars, huge map and marathon speed). Besides the population growth the reconstructed phase space is shown.

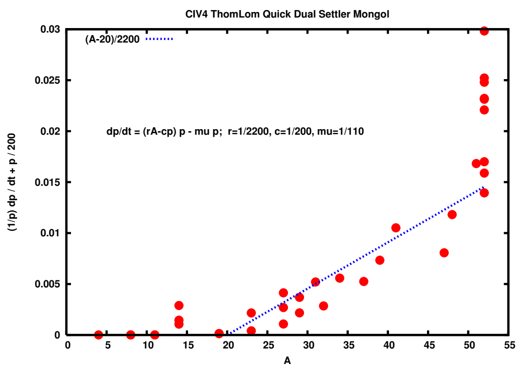

Figure 9: Population dynamics in the CIV IV game ThomLom (Dual (mini) world, Settler level, no wars, quick speed). The logarithmic growth rate of the population modified with a preset term is plotted against the area of the ThomLom civilization. Parameters of the monogamic demographic equation (16) with $\alpha = 0$ are obtained from fitting the linear range.

4.3 Case study II: CIV games

Since statistical data from history are rare and insecure, it is interesting to analyze simulation data. In the Tomgis game with CIV-4 (settler level, huge map, marathon speed and no wars) we have monitored the population, its growth and the area of the selected civilization. Fig.8 presents the population dynamics as a function of time (left) and in the phase space (right).

In Fig.9 the parameters of the monogamic demographic equations are obtained for a given game in a minimal (dual) world with no wars and quick evolution speed. The logarithmic growth rate, the basic demographic variable is investigated as a function of the area[10]. The linear part of the evolution fits well the monogamic equation (16) with $\alpha = 0$. In the late phase the area saturates, because no more virgin land is available. A further growth in A could only be achieved by war, which was not practiced in this particular game. However the population growth continues, the evolution is still well ahead of the saturation point.

5 Areal dynamics: war and peace

Of course it is impossible to grow infinitely in a finite system. As the virgin lands[11] full with resources and free for colonization become sparse and eventually diminish, the dynamics of area growth changes. In this process warlike conflicts take over the place of colonization and cultivation; this probably has happened first between the neolithic and the classical antique era. The development of technology, science and a general civil culture may ease this process time to time making available an intensification of resource usage; therefore wars occur in several cycles at essentially unpredictable times. At the same time the evolution of technology, area and population drives the trend for more and more devastatingi wars. Beyond a certain level war-devastation may be disadvantageous even for the victors; this way a new regulating mechanism emerges through the rational deliberation of war and peace. In this section we model this process by simple mathematical means, assuming that the aim of deliberation is connected to the wish to increase the own resource usage, $rA = rL^2$.

Due to a war the area colonizable by a given civilization may be increased, but the reduction in the population growth rate is unavoidable due to the very efforts necessary for that war. A balance between these two processes may lead here to a stationary state, statistically characteristic for a given era and larger area. For the sake of a simplified analysis we do not consider the interaction (war) of civilizations pairwise, but rather treat this phenomenon in the framework of a *mean field* model: a single civilization is regarded against an average environment of other (hostile, neutral or cooperating) civilizations.

According to our basic assumption the main sacrifice due to conducting a war is not the direct death of participating (or in the case of civilians just misfortunate) people, but the decrease of the growth factor: we assume that the factor rL^2 is reduced to $rL^2 - hv$, denoting by h the war weariness per capita and by v the average victory factor of the contestants against us. The latter is conjectured to be proportional to the average population of the "enemies", $v = \alpha\langle p\rangle$. The demography equation (here we consider the full polygamic version, but this makes no qualitative difference) is modified during wars:

$$\dot{p} = -cp^3 + (rL^2 - hv)p^2 - \mu p \tag{21}$$

[10] CIV-4 makes statistics of population and area among others.

[11] Gardens of Eden

The saturation point in peace, p_S is endangered, unless the risk hv is not negligible compared to our resources rL^2: we are a big empire, an international superpower or technologically so advanced that our foes even cannot steal our knowledge. In the case when $(rL^2 - hv) \leq 2\sqrt{\mu c}$ our own civilization may slide into an extinction path. In such cases it would be a catastrophe to initiate a war from our side[12].

Yet there were (and are) wars. We would like to know whether is it sometimes rational to initiate a war (at least in the CIV simulation). On the other hand is the principal pacifism an utopistic dream or it can be a result of rational and utilitarian deliberation? Is there any difference from this viewpoint between small and poor (small rL^2) and large and rich (large rL^2) civilizations? It is remarkable that the following, hopelessly oversimplified mathematical model gives some answers to the above questions at all. It is a separate question how much one likes or dislikes these answers.

The result of a war, besides the aching loss in the reproduction strength, could also be a gain in the area under control, with a certain probability. Let the probability of victory, due to which an increase in the linear size occurs, $L \to L + a$ (a stands for annection), be V, the probability of an approximately same areal loss ($L \to L - a$) on the other hand D (for defeat). If there were no other option, like a compromise peace treaty, which reestablishes the status quo before the war and usually triggered by war weariness, internal rebellion or financial crash, then $V + D = 1$, otherwise $V + D < 1$. There is no way to peace, peace is the way. The probability of this third way is denoted by P. In any case $V + D + P = 1$.

Based upon this the condition for avoiding the extinction path due to a war weights the results with the respective probabilities:

$$\big(r(L+a)^2 - hv\big)\,V + \big(r(L-a)^2 - hv\big)\,D + \big(rL^2\big)\,P > 2\sqrt{\mu c}. \qquad (22)$$

Executing the quadrations and subtracting rL^2 from both sides of the inequality we arrive at the rational condition for initiating a war:

$$2rLa(V-D) + (ra^2 - hv)(V+D) > 2\sqrt{\mu c} - rL^2. \qquad (23)$$

The expression on the right hand side of this inequality is negative, since before the war the initiator civilization was on a population growth track: the p_S saturation point existed. In order to reduce the triggering off a war in the model, i.e. considering the basically peaceful nature of the human psychology, we consider a stronger condition for a civilization to become interested in unleashing a war with the others:

$$2rLa(V-D) + (ra^2 - hv)(V+D) > 0 > 2\sqrt{\mu c} - rL^2. \qquad (24)$$

[12]Remember the fortune-telling in Delphi: "If you start this war a great empire shall be ruined".

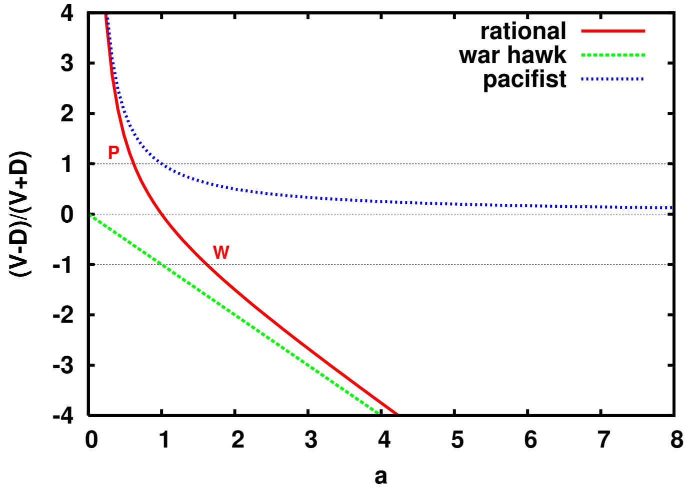

Figure 10: The rational, the pacifist and the hawk deliberation lines for a civilization of size L for a war initiated to gain territorial control against an average strength environment as a function of the expected gain in linear size, a. The actual curve shown was made with the parameters $2L = 1$ and $hv = r$.

From this condition the following "rational" deliberation can be derived: it is promising to risk a conflict in the hope of achieving a growth like $L \to L + a$, if the odds for victory V and defeat D do satisfy the following inequality:

$$\frac{V-D}{V+D} > \frac{1}{2L}\left(\frac{hv}{ra} - a\right). \tag{25}$$

This formula has some interesting lessons. Let us start with the simplest: if $hv = ra^2$, i.e. the expected loss is just equal to the expected gain in our growth factor, then it should be $V > D$, i.e. the victory shall be more probable than defeat.

The decision about war is often made then, when the "good old status quo" cannot be kept any more. If peace is no alternative, then $P = 0$, therefore $V + D = 1$. But this is only a special case of the following analysis.

Whenever peace is also an alternative, $P > 0$ and hence $V + D < 1$, the left hand side of the inequality (25) is larger, the condition is lighter. If the expected loss is huge, $hv \gg ra^2$, then the requirement on $V - D$ is stronger; a greater chance of victory is demanded for starting a war on a rational basis. In any case the odds for victory and defeat both are numbers between zero and one, therefore $V - D$ falls between -1 and 1. Whenever $V + D$ is close to 1, i.e. there is little chance for keeping the area, it will either be increased or decreased, then the line $V = 1$, $D = 0$ means the sure victory, and the line $V = 0$, $D = 1$ the certain defeat. Beyond these lines, independently of the victory chance, exist

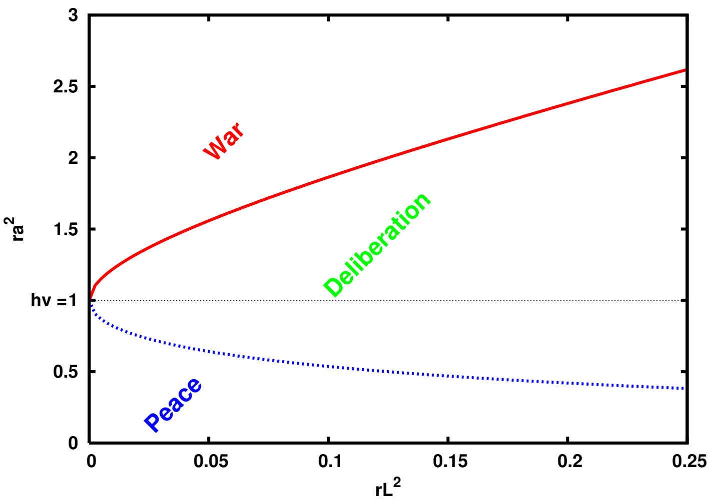

Figure 11: The regions of rational peace and war on the map of the present and gainable resources (rL^2, ra^2). The region for deliberation opens up between the lines $(V-D)/(V+D) = -1$ and $(V-D)/(V+D) = +1$: this rapidly grows with rL^2. This figure is made choosing $hv = 1$.

the regions of lasting peace and unresting wars. This border in the rational deliberation is indicated in Fig.10 by the red line.

For the sake of comparison we also show the line of the extreme pacifist consideration, reducing the factor $hv/ra - a$ to hv/ra (i.e. neglecting the lure of a possible gain in size), and the line of the war hawks, seeing only the possible area gain and hence counting with $-a$ instead of $hv/ra - a$ in the formula. These strategies are indicated by blue and green lines, respectively.

Figure 11 circumlines the regions of (rationally) sure peace and war on the map spanned by the resources risked to gain or loose, ra^2, and by the starting resource strength of the civilization, rL^2. Peace rules in the bottom, war in the top region. It is easy to inspect that around the trivial midline, $ra^2 = hv$, opens up the region of rational deliberation about war or peace. The lower border of this region is characterized by $(V-D)/(V+D) = -1$, the upper one by $(V-D)/(V+D) = +1$. The asymptotic lines for large rL^2, relevant for great and rich civilizations, are the lines of pacifists, $ra^2 = (hv)^2/4rL^2$, and war hawks, $ra^2 = 4rL^2$, respectively.

It is clear that even in those times, when keeping the status quo is not an alternative, i.e. $V + D = 1$ and $P = 0$, the freedom in deliberation for great and rich civilizations grows rapidly with respect to the meridian of equal gain and loss, $ra^2 = hv$. It seems that developing in economic and population strength increases the chances both for world wars and for the world peace.

6 Resource improvement: technological development

War becomes less and less an alternative by the diminishing of gainable resources. In such cases "smart" civilizations try to increase the factor r instead (or besides) L. They try to intensify the use of resources on their given area by developing scientific research and technological applications. In a broad sense social and cultural practice, which increase the r factor, are also technological; such are in particular state forms, legal inventions, or expansive religious ideologies. As a side effect the death rate, μ may be reduced (inventions in medicine, public health and hygienics), and new areas can be included in resource usage, regarded as worthless before the progress (e.g. oil fields, sea plantages, geothermic heat). Such steps increase L without reducing the area of other civilizations.

Furthermore if science and technology develop, so does war technology, so the quantity $V-D$ can also be increased by scientific research. Exactly by promising this may gain the developer some support from the representatives of the community (in the rule from sovereigns or other influential and rich people). On the other hand a runaway technological progress, due to the pollution of environment by increased waste production and due to the too rapid exhaustion of natural resources, the effective (usable) rL^2 factor may even be smashed down in a sudden crisis[13]. Of course, it is not an alternative to give up and stop searching for further possibilities to increase the growth. But "sustainable growth" may be just a dream without considering new ways to increase rL^2; in the very end by dropping the restrictions posed by a lonely planet, Earth.

7 Using the resources of Others: trade

What to do when neither area annecting wars nor the scientific-technological development does offer us a rapid and sustainable growth? The factor L (or A) can still be increased virtually, if neighboring or farer away civilizations incline to share with us their resources: this act at the same time leads to the occurrence of new professions (traders, road architects, traveling agents), which in turn is synergic for increasing the resource usage factor r. The new channels of luxury, of course, eventually also lead to an increased consumption and by spreading out in the population they may slightly increase the factor c, too. The latter process is, however, slower, its effect evolves on a longer time scale.

The development of the infrastructure, transport and communication, leads to a better usage of the own resource area as well as to the usage of an enlarged area (with regions standing under the domination of another civilizations). Last but not least it leads to a more effective usage of armies, and this way may increase

[13] Even Greens might be right in some respect.

$V - D$ against such follies, who do not invest enough in the development of trade.

8 Spread of strategies: the culture

Among all possible ways of progress the fastest seems to be the takeover of strategies developed or discovered by others. It depends on the cultural strength of a given civilization, on its cultural distance from the other one, that from whom and which new development can be taken over to increase r or effectively L, perhaps to decrease μ. A very much backdoor culture may fail to naturalize a developed social or material technology, e.g. due to a lack of properly educated people. On the other hand the own development, past successes in compromise finding peace diplomacy, the luxurious consumption including the "high culture", and the public satisfaction with the religious and education system and in general with social emergence possibilities all increase the cultural strength. Briefly, it is the best to be the best, but is not very much rewarding to be even better than that.

9 Cycles: are they real?

One meets quite frequently with the idea of cycles, eras of rise and fall in the life of civilizations. Toynbee, Russian authors in particular Kondratyev, but also medieval Arabic scripts deal with this motive. In a mathematical description cycles rely on a positive and negative coupling between two mean variables: one increases the other, but the other decreases the first. In this case a limiting cycle describes the attracting dynamics in place of some fixed points. In my view these effects do not play a role on the long term in the dynamics of civilizations in several hundreds of years. However, since this idea plays an important role in the human culture, we have to list a brief review:

- The Kondratyev cycles describe economical and financial crises on the time scale of 10 - 30 years.
- Ibn Khadún had discovered cyclic behavior in the mutual dynamics of settled, state holding agrarian civilizations and their environing barbaristic tribes in the twelfth century. When the civilized state weakens (due to the unavoidable growth of corruption and the corresponding expenses on the state budget, and due to the growing luxurious consumption of the burocratic ruling class), it will be militarily defeated by the nomads, who constitute the new ruling class. The characteristic time for a cycle is about 60 years, two-three human generations.
- Turchin points out secular cycles in the development of traditional agrarian societies observing and mathematically modeling financial and political power crises in 100-150 years. Due to Turchin the reason being the

coexistence of a positive and a negative feed back: the growth of the population urges the growth of the state organization, but the expanding state becomes more and more expensive leading unavoidable to a crisis with decreasing wages and increasing prices and this effect reduces the population. This process is escorted by civil wars and other aggressive acts triggered by the increased population pressure and price inflation.

10 Shocking events: natural and social catastrophes

In the course of history sometimes the random fluctuations exceed the limit under which they can be treated as a background noise. Among these such events may occur which influence the fate of one or the other civilization essentially; they cause a sudden, hardly tolerable change in the parameters of the basic dynamical equations. We call such situations "shocking events", a concept akin to Turchin's resetting event, but it can be good as well as evil, development boosting as well as destructive.

Such an event can be a major natural catastrophe (volcano eruption, tsunami, major earthquake) or a sudden change in the climate (droughts or floods). Changes of human origin also can cause a sudden change in the parameters; the chronicles rather record the bad ones (but among these rather those which could have been tolerated by the civilization). The legend of Atlantis or the biblical story of the Big Flush show that the effect of such events made people think already in ancient times.

There are several examples of resetting (population degrading) events from the Middle Ages: The Mongolian Conquest for East-Europe or the Black Death (pest) for West-Europe. The characteristic problem for the long term development is the sudden decrement of available resources (cultivable land, working power).

In our simple model the effect of a shocking event can be presented transparently by an example which has been occurred in the modern Hungarian history. The shocking event is the coincident and essential drop in the population, p and resource area A: $p' = 2p/3$, $A' = A/3$. Let us consider that this happens in the modern times, when the increase of the area A due to emigration and colonization is almost done, so this is not an option. At the same time the medicine and the public health is on a high level, so the approximation $\mu \approx 0$ can be made. This is a simplified scenario.

The civilization (the social system) was functioning before the event in a stable saturation point with a population of $p_S = 15$ M (million) and an area of roughly $A = 0.3 Mkm^2$ (three hundred million square kilometer). The area is reduced suddenly to its one third and the population to its two third by the shocking event. Using the above mentioned data the ratio of resource usage to the consumption/corruption loss was $r/c = 50$ km^{-2}. In the pure monogamic ($\alpha = 0$) demographic model

$$\dot{p} = -cp(p - p_S), \tag{26}$$

the population saturation point was at $p_S = (rA - \mu)/c = 15$ M, while $\mu = 0$. By these numbers the sudden drop of the population does not cause an immediate extinction, the system is expected to develop into the new saturation point, from $p' = 10$ M to $p'_S = 5$ M, within a given time, since rA/c has been reduced to its one third. What damage was not done immediately by the war, that will be done by the reduced dynamical parameter due to the disclosure of resources. At least this should be expected by a simplified rational consideration.

But societies react to the shock by unusual efforts. Instead of waiting for the new, much lower saturation point to occur, the civilization tries to lift the saturation point close to the actual population number. The people are emotionally driven to double the factor r in order to reach $p'_S = 10$ M. This, i.e. $r' = 2r$, is in principle achievable; best by a multiple synergic strategy: for example a +26% increment may be achieved in r by a general reform of the political and juristic system, another +26% improvement due to a more intensified public education and finally one more +26% growth due to the development of trade and financial system. This altogether amounts to $r' = 1.26^3 r \approx 2r$.

The parameter μ in a real situation is of course not zero, mortality, aging of the society, emigration all contribute. Its value, however, in successful modern civilizations is relatively low, the ratio of μ/c relative to rA/c may be around $0.01 - 0.1$. To be set to a road to extinction in the monogamic demography model therefore a reduction to $1 - 10\%$ of the original population belongs. Counting with one and a half emigrants in the above example the required improvement in the resource usage efficiency is even higher than considered naively above, $r'/r = 11.5/5 = 2.3$. Defactorizing it to three different synergic strategies it means a 32% increment on each subfactors.

The above analysis, however historically unrealistic, leads to different conclusions in the polygamic demography model. The situation before the shocking event is characterized by the numbers $p_S = 15$ M, $p_M = 5$ M and $A = 0.3 Mkm^2$. The dynamical parameters extracted from this are

$$\begin{aligned} \frac{rA}{c} &= \quad p_S + p_M \quad = 20M, \\ \frac{\mu}{c} &= \quad p_S \cdot p_M \quad = 75M^2 \end{aligned} \tag{27}$$

The minimal resource usage efficiency necessary to avoid extinction in this case is given by

$$\left(\frac{rA}{2c} \right)_{\min} = \sqrt{75} M \approx 8.66 M. \tag{28}$$

This value was $rA/2c = 10$ M before the shock, it has been reduced to $10/3 \approx 3.33$ M by restricting the area A to one third. The latter value is below the extinction threshold. The survival has now two conditions: i) the value of $rA/2c$ has to be increased at least to about 8.7 M and ii) it has to be arranged that the Malthus point, p'_M, would not exceed the actual population after the shock, $p' = 10$ M. The minimal increase in efficiency must be $r'/r = 8.66/3.33 \approx 2.6$ for a new saturation point to exist at all. If one wants this saturation point at $p'_S = 10$ M, then considering the same μ/c ratio the new Malthus point becomes $p'_M = 75/10M = 7.5M$. In order to reach such a stage an increase of $r'/r = 3(9/10) = 2.7$ is necessary. For a total compensation of the area loss effect on the other hand an even higher ratio, $r'/r = 3$ would be necessary. And after all these considerations one should not forget that meanwhile the r factors of the concurring civilizations are also have been increased.

A probably more realistic estimate can be gained if we assume $p_M = 0.02$ M besides $p_S = 15$ M. It means that below a population of twenty thousands the extinction path would be followed even with the benign parameters before the shock. In this case $rA/c = 15.02$ M and $\mu/c = 0.3M^2$. The minimal resource ratio is 0.5477 M from the formula $(rA_{\min}/2c) = \sqrt{\mu/c}$, i.e. about half a million people. This number is three and a half per cent of the original population, in this case an area loss of as large as 96% might be survived. These data are close to the case discussed in the $\mu/c = 0$ approximation, so for a new saturation point of $p'_S = 10$ M after a reduction of the area to its one third the resource efficiency must be doubled: $r'/r = 2$.

11 Slightly polygamic demography

In a realistic human society neither the purely monogamic nor the extreme polygamic reproduction strategy dominates. A more flexible demography equation introduces an extra power α between zero and one:

$$\dot{p} = (rA - cp)p^{1+\alpha} - \mu p. \tag{29}$$

Modern societies may realize a small value, say $\alpha = 0.1$. This shifts the saturation point only slightly with respect to the pure monogamic case:

$$p_S = \frac{rA - \mu}{c} + \alpha \frac{\mu}{c} \ln \frac{rA - \mu}{c} + \mathcal{O}(\alpha^2). \tag{30}$$

The Malthus point approaches zero, its order of magnitude is $\alpha \ln \alpha$. The condition for growth is given by

$$rA \;>\; (1+\alpha)\, \mu^{\frac{1}{1+\alpha}} \, \left(\frac{c}{\alpha}\right)^{\frac{\alpha}{1+\alpha}} . \tag{31}$$

This formula leads to $rA > \mu$ in the monogamic ($\alpha = 0$) case and to $rA > 2\sqrt{c\mu}$ in the extreme polygamic case ($\alpha = 1$).

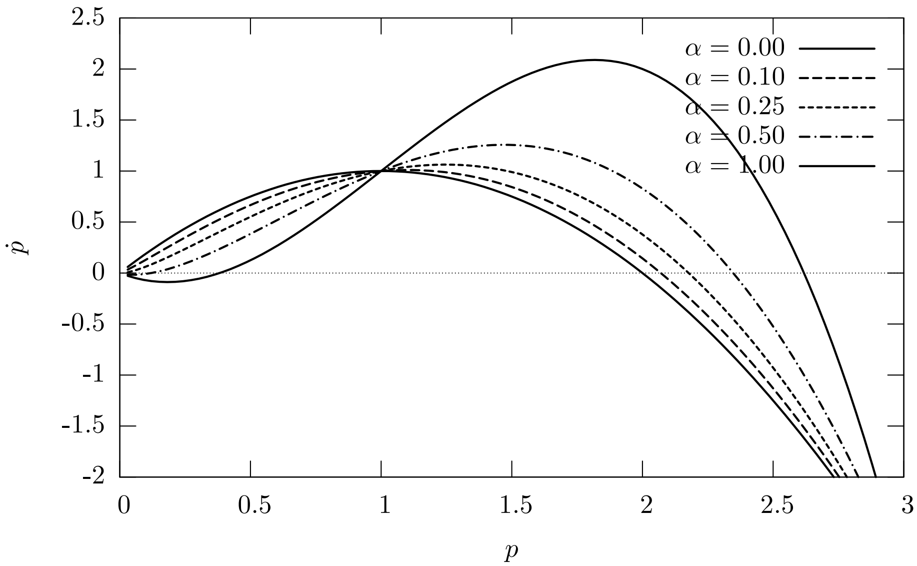

Figure 12: The phase space of the population dynamics for different α polygamy indices (in scaled units) to the left, an enlargement of the $\alpha = 0.1$ curve in the neighborhood of the Malthus point to the right.

Figure 12 plots the minimal threshold value of the resource to consumption ratio, rA/c, necessary for a sustainable growth as a function of the polygamy index α, for several μ/c ratios.

12 Diffusion of population

A possible diffusion model which establishes the above mean field dynamics can be constructed. In this model the population density is a time-dependent, two-dimensional field, $\Pi(x, y, t)$ satisfying a linear diffusion equation,

$$\dot{\Pi} \;=\; \lambda(A, p)\, \Delta\Pi \;+\; \varphi(A, p)\, \Pi \tag{32}$$

with $\Delta = \partial_x^2 + \partial_y^2$ being the two-dimensional Laplace operator. The diffusion constant $\lambda(A, p)$ and the growth factor $\varphi(A, p)$ depending on the area $A(t)$ and on the population $p(t)$, will be specified as follows.

The population is defined by the integral of the population density over the whole (in the theory infinite) world:

$$p(t) = \iint dxdy\; \Pi(x, y, t). \tag{33}$$

The area of influence, A, will be defined on the basis of the solution of the diffusion equation (32). The solution normalized according to eq.(33) is given by

$$\Pi(x,y,t) \;=\; \frac{p}{A}\, e^{-\pi(x^2+y^2)/A}. \tag{34}$$

Here the area parameter is bound to the width area of the Gaussian solution, it is the expectation value of the area of a circle weighted by the population density:

$$A = \langle \pi(x^2+y^2) \rangle. \tag{35}$$

Substituting the solution (34) into the diffusion equation (32) one arrives at

$$\begin{aligned} \dot{p} &= \varphi(A,p)\, p \\ \dot{A} &= \lambda(A,p). \end{aligned} \tag{36}$$

Here the functions $\varphi(A,p)$ and $\lambda(A,p)$ can be set to those treated in the mean field approximation. In particular the annual growth factor $\dot{p}/p$ follows the population dynamics behavior of the monogamic scenario

$$\varphi(A,p) = rA - cp - \mu, \tag{37}$$

and the diffusion coefficient is set to

$$\lambda(A,p) = \frac{\kappa}{2\pi}\left(\frac{p}{A} - m\right)\sqrt{A}. \tag{38}$$

For the purpose of the further analysis we express the area of influence defined as the area where the population density exceeds a forefixed threshold value, w (i.e. $\Pi(x,y,t) \geq w$ for $\pi(x^2+y^2) \leq A_w$). For the Gaussian solution this is given as

$$A_w = A \ln\left(\frac{p}{wA}\right). \tag{39}$$

The main hypothesis of the diffusive model presented in this section is that *the migration and colonization of the civilization tries to maximize this area*, since this optimizes the growth of the civilization at a given population number. The position and the value of the maximal area can best be expressed by using the square width area, $A(t)$, defined in eq.(35). The position of the maximum of A_w at a fixed p in eq.(39) is given by

$$A_{w,\max} = \frac{p}{we} = A_{\max} \tag{40}$$

with $e \approx 2.72$ being the Euler number. It is intriguing that this maximal area value is proportional to the population; this phenomenon has been observed in several CIV games.

It is finally interesting to compare the mean field dynamics of the controlled area, comprised in eqs.(36, 38) with the area maximizing hypothesis at the fixed population density threshold. The evolution equation for the Gaussian area $A(t)$, which follows from these two equations,

$$\dot{A} = 2\kappa\left(\frac{p}{A} - m\right)\sqrt{A}, \tag{41}$$

can be re-written in terms of this maximal area if choosing $m = we$ as the migration threshold in mean field dynamics:

$$\dot{A} = \frac{2\kappa}{\sqrt{A}} \left(A_{w,\max} - A \right). \tag{42}$$

The dynamics described by eq.(42) not only stabilizes A at $A_{w,\max}(p)$ for a given population p, but it does so with a rate diminishing by the linear size of the empire, $\lambda \propto 1/\sqrt{A}$. This reflects the fact that late colonialist must migrate a longer distance before finding virgin areas to civilize. This effect is similar to the well-known phenomenon of the "stretching of the train" from studies of military expeditions, in particular in the antique (e.g. Alexander the Great). The factor κ may be increased by better roads and other technological developments boosting the speed of traffic.

13 Summary: solving or making equations?

The traditional attitude towards mathematical models in sciences like theoretical physics is solving equations. The sensitivity to initial conditions and parameters, mostly assumed to be constants, is a primary goal of the studies in order to gain some control and insight about phenomena to govern. In the mathematical models of human relations and large systems including the conscious human component rather the equations themselves are to be found.

Two characteristic features can be formulated: i) It can never be assumed that a parameter is less changeable then the so called variable and ii) these parameters depend on the other variables in a way which may seem to be purposeful: human societies after a while select strategies to optimize important factors. And, like in the case of the evolution of competing species in the drama of life, those are more likely to survive and prosper longer, whose goals include the growth of the population, or any important variable which is synergic to the growth of the population.

The simple model presented in this paper has therefore no request to be final. However, the basic dynamics of integral quantities of a distributed population, which is handling under the influence of random forces but communicates her experiences internally, in the way of the past human history contains a "Malthusian" element, therefore a demographic equation must be part of any mathematical model of the rise and fall of civilizations. Furthermore it arises quite naturally from a view of human migration and colonization as a locally (in the population density) linear process that the area of influence of civilizations is the second main factor describing the dynamics. This has a very simple mathematical ground: a Gaussian population density has two independent parameters, its integral norm and its width, and due to the central limit theorem a large number of random effects are bound to lead to a Gaussian distribution. Exception from this rule are non-equilibrium, non-stationary and open systems, and non-locally interacting networks.

The latter is rather typical for the evolution of the parameters of the basic dynamical equations, like the resource usage efficiency r, the braking effect of consumption on the population growth, c, the mortality μ, the probabilities of victory and defeat in wars for area and resource increase, because these are the objects of human strategies. This is also the ultimate sense to play with mathematical models and simulations of the civilizatory evolution and dynamics: we may – at the end – extract deals for our present and future strategies to achieve sustained growth for our own civilization.