Dark Energy from Bohmian Gravitational Binding

Einstein's Schrödinger Equation

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Naturalness: where are you?

The "naturalness" problem:

Why is the cosmological constant so small?

in natural units: $\mathbf{L}_P^2 \cdot \Lambda \approx 10^{-122}$

Outline

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Schrödinger eq. in magnitude-phase representation

$$-\frac{\hbar^2}{2m}\nabla^2\varphi + V(x)\varphi = i\hbar\frac{\partial}{\partial t}\varphi \tag{1}$$

General radial form for the complex wave function:

$$\varphi = R\, e^{\frac{i}{\hbar}\alpha}$$

Classical action and momentum

$$\frac{\partial\alpha}{\partial t} = -E, \qquad \nabla\alpha = P \tag{2}$$

Logarithmic Derivatives

$$\frac{\partial}{\partial t}\varphi = \left(\frac{1}{R}\frac{\partial R}{\partial t} - \frac{i}{\hbar}E\right)\varphi, \qquad \nabla\varphi = \left(\frac{1}{R}\nabla R + \frac{i}{\hbar}P\right)\varphi \tag{3}$$

Laplacian

$$\nabla^2\varphi = \left[\nabla\left(\frac{\nabla R}{R} + \frac{i}{\hbar}P\right) + \left(\frac{\nabla R}{R} + \frac{i}{\hbar}P\right)^2\right]\varphi \tag{4}$$

Real and Imaginary Part

$$E = V - \frac{\hbar^2}{2m}\left[\nabla\frac{\nabla R}{R} + \left(\frac{\nabla R}{R}\right)^2 - \frac{P^2}{\hbar^2}\right] \tag{5}$$

$$\frac{i\hbar}{R}\frac{\partial R}{\partial t} = -\frac{\hbar^2}{2m}\frac{i}{\hbar}\left[\nabla P + \frac{2}{R}P\cdot\nabla R\right] \tag{6}$$

Interpretation

Re: Energy = **Classical** + **Quantum** (Bohm potential)

$$E = \boxed{\frac{P^2}{2m} + V} - \boxed{\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}} \tag{7}$$

Im: Mass density current continuity

$$m\frac{\partial R^2}{\partial t} + \nabla\left(R^2\, P\right) = 0 \tag{8}$$

Action Principle

Variational Principle behind the Schrödinger equation

$$\mathfrak{S} = \int \left(\frac{\partial S}{\partial t} + \frac{|\nabla S|^2}{2m} + V \right) |\varphi|^2 \, d^3x \, dt \tag{9}$$

"Boltzmannian" eikonal ansatz: $S = \frac{\hbar}{i} \ln \varphi$ Using this ansatz:

$$\mathfrak{S} = \int \left[\frac{\hbar}{i} \varphi^* \frac{\partial \varphi}{\partial t} + \frac{\hbar^2}{2m} \nabla \varphi^* \cdot \nabla \varphi + V \varphi^* \varphi \right] d^3x \, dt \tag{10}$$

Variation against φ^* delivers

$$\frac{\delta \mathfrak{S}}{\delta \varphi^*} = \frac{\hbar}{i} \frac{\partial \varphi}{\partial t} - \frac{\hbar^2}{2m} \nabla^2 \varphi + V \varphi = 0 \tag{11}$$

Action with Magnitude - Phase Variables

Split to Quantum + Classical parts

$$\mathfrak{S} = \int \left[\frac{\hbar^2}{2m} (\nabla R)^2 + R^2 \left(\frac{(\nabla\alpha)^2}{2m} + V + \frac{\partial\alpha}{\partial t} \right) \right] d^3x\, dt \qquad (12)$$

Structure of Quantum Principle:

$$\mathfrak{S} = \hbar^2(\text{ quantum kinetic }) + R^2(\text{ classically zero })$$

Path integral, tunneling: $S = \alpha - i\hbar \ln R$

Outline

Quantum Lagrangian

Lagrange density

$$\mathcal{L} = \frac{1}{2}\partial_i\psi^* \, \partial^i\psi \; - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2 \psi^* \, \psi. \tag{13}$$

Action and other conventions

$$\mathfrak{S} = \int \mathcal{L} \, d^4x \tag{14}$$

with $dx^i = (cdt, d\vec{r})$.
Physical units $[\mathcal{L}] =$ energy density / c = $[mc/L^3]$.

Madelung ansatz

Complex scalar field `(related to the wave function)`

$$\psi = \frac{\hbar}{\sqrt{mc}}\, R\, e^{i\alpha/\hbar}. \tag{15}$$

Here α is only the (real) classical action. Units of R from

$$(mc/\hbar)^2\ \psi^*\psi\ =\ mc\, R^2 \tag{16}$$

which is part of $\mathcal{L}$; it follows that R^2 is a **number density**.
Compare this with Maupertuis action for a classical mass point:

$$-\frac{1}{2}\int\left(\int mc^2 R^2 d^3x\right) dt\ =\ -\int mc^2 d\tau \tag{17}$$

It follows a normalization $\int R^2\, d^3x\ =\ 2$ in the comoving frame.

Derivative

First derivative of ψ:

$$\partial_i \psi = \left(\frac{\partial_i R}{R} + \frac{i}{\hbar} \partial_i \alpha \right) \psi \tag{18}$$

The derivative of the classical action is a classical momentum:

$$P_i = \partial_i \alpha, \qquad u_i = P_i/(mc). \tag{19}$$

Quantum Action

$$\mathfrak{S} = \frac{\hbar^2}{2mc} \int \left[\partial_i R \, \partial^i R + \frac{R^2}{\hbar^2} \left(\partial_i \alpha \, \partial^i \alpha - (mc)^2 \right) \right] d^4x \qquad (20)$$

Rewritten as a sum of a quantum and a classical part:

$$\mathfrak{S} = \int \left[\boxed{\frac{\hbar^2}{2mc} \partial_i R \, \partial^i R} + \boxed{\frac{R^2}{2mc} \left(P_i P^i - (mc)^2 \right)} \right] d^4x \qquad (21)$$

U(1) Noether charge

$$J^k = \frac{i}{2\hbar}\left(\psi\partial^k\psi^* - \psi^*\partial^k\psi\right) = \frac{1}{mc}R^2P^k = R^2u^k \tag{22}$$

is a number density 4-current $R^2u^k = \rho u^k$.
Conserved by variation of $\mathfrak{S}$ wrsp. α:

$$\frac{\delta\mathfrak{S}}{\delta\alpha} = -\partial_k\left(\frac{1}{mc}R^2\partial^k\alpha\right) = 0 \tag{23}$$

Facit:

$$\frac{\delta\mathfrak{S}}{\delta\alpha} = \frac{\delta S_{\text{quantum}}}{\delta S_{\text{classical}}} = -\partial_k J^k = -\partial_k(\rho u^k) = 0. \tag{24}$$

Off Mass Shell

Variation against R delivers

$$\frac{\delta \mathfrak{S}}{\delta R} = -\frac{\hbar^2}{mc}\Box R + \frac{R}{mc}\left(P_i P^i - (mc)^2\right) = 0. \tag{25}$$

Off-mass shell dispersion relation for the classical 4-momentum

$$\boxed{P_i P^i - (mc)^2 = \hbar^2 \frac{\Box R}{R}} \tag{26}$$

Metric view:

$$g_{ij} u^i u^j = 1 + \left(\frac{\hbar}{mc}\right)^2 \frac{\Box R}{R} \tag{27}$$

Note: this is a `Compton wavelength scaled,` **locally Lorentzian** `spacetime metric.`

Energy-Momentum Tensor I (textbook)

Derivation I: using ψ and ψ^*. (... as in textbooks)

$$\Pi_k = \frac{\delta \mathcal{L}}{\delta \partial^k \psi} = \frac{1}{2} \partial_k \psi^* \tag{28}$$

$$T_{ij} = \Pi_i \partial_j \psi + \Pi_i^* \partial_j \psi^* - g_{ij}\, \mathcal{L}. \tag{29}$$

in terms of R and α:

$$T_{ij} = mcR^2\, w_{ij} + \frac{\hbar^2}{mc} U_{ij},$$
$$w_{ij} = u_i u_j - \frac{1}{2} g_{ij}\,(u_k u^k - 1),$$
$$U_{ij} = \partial_i R\, \partial_j R - \frac{1}{2} g_{ij}\, \partial_k R\, \partial^k R. \tag{30}$$

Energy-Momentum Tensor I (textbook)

Using the off-mass-shell relation (26) leads to:

$$T_{ij} = mcR^2 u_i u_j + \frac{\hbar^2}{mc}\left(\partial_i R\, \partial_j R - \frac{1}{2} g_{ij}\left(\partial_k R \partial^k R + R\Box R\right)\right) \tag{31}$$

Here the $\hbar^2(\,)$ part is the quantum contribution, the rest is just **dust**.

Energy-Momentum Tensor II (Takabayashi)

Take the derivative of the off-mass-shell eqation (27):

$$\frac{R^2}{2}\,\partial_i\left[u_k u^k - 1 - \frac{\hbar^2}{(mc)^2}\frac{\Box R}{R}\right] = 0 \tag{32}$$

Use Compton wavelength $L_C = \hbar/mc$ and expand:

$$R^2 u^k \partial_i u_k - \frac{1}{2} L_C^2 R^2 \partial_i\left(\frac{\Box R}{R}\right) = 0 \tag{33}$$

Energy-Momentum Tensor II (Takabayashi)

The Madelung fluid is irrotational:

$$\partial_i u_k = \frac{1}{mc}\partial_i\partial_k\alpha = \frac{1}{mc}\partial_k\partial_i\alpha = \partial_k u_i \qquad (34)$$

Therefore

$$R^2 u^k \partial_i u_k = R^2 u^k \partial_k u_i = \partial_k(R^2 u^k u_i) - u_i\partial_k(R^2 u^k) \qquad (35)$$

and due to continuity the last term vanishes.

Energy-Momentum Tensor II

By these manipulations we obtain

$$\partial_k \left(R^2 u^k u_i \right) = \frac{1}{2} L_C^2 R^2 \partial_i \left(\frac{\Box R}{R} \right) \tag{36}$$

Further use of the Leibniz rule leads to

$$\begin{aligned} R^2 \partial_i \left(\frac{\Box R}{R} \right) &= R \Box \partial_i R - \partial_i R \, \Box R \\ &= \partial^k \left(R \partial_k \partial_i R - \partial_k R \, \partial_i R \right) \end{aligned} \tag{37}$$

Energy-Momentum Tensor II

This reveals a vanishing divergence of the Bohm-Takabayashi tensor

$$\mathcal{T}_{ij} = mcR^2 u_i u_j - \frac{\hbar^2}{2mc}\left(R\partial_i\partial_j R - \partial_i R\,\partial_j R\right) \tag{38}$$

It differs from the Klein-Gordon one (31) by

$$\Delta_{ij} = T_{ij} - \mathcal{T}_{ij} = \frac{\hbar^2}{2mc}\Big(\partial_i R\partial_j R + R\partial_i\partial_j R - g_{ij}(\partial_k R\partial^k R + R\Box R)\Big) \tag{39}$$

Difference

We note that

$$\left(g_{ij}\Box - \partial_i\partial_j\right)\frac{R^2}{2} = g_{ij}(\partial_k R\partial^k R + R\Box R) - \partial_i R\partial_j R - R\partial_i\partial_j R \quad (40)$$

One realizes that

$$\Delta_{ij} = \frac{\hbar^2}{4mc}\left(\partial_i\partial_j - g_{ij}\Box\right)R^2 \quad (41)$$

`has a vanishing divergence.`

Note: $\Delta_{ij} = \partial^a f_{aij}$ with $\boxed{f_{aij} = \frac{\hbar^2 R}{2mc}\left(g_{ai}\partial_j R - g_{ij}\partial_a R\right).}$

Full Quantum Energy-Momentum Tensor III

It can be derived in two ways: from Klein-Gordon Lagrangian and from Madelung fluid hydrodynamics.
They differ in a tensor part with *vanishing divergence*.
The general tensor contains parameter μ multiplying the divergenceless part:

$$\mathfrak{T}_{ij} = mcR^2 u_i u_j + \mathfrak{U}_{ij} + \mu \Delta_{ij} \tag{42}$$

with the general Bohm-Takabayashi term

$$\mathfrak{U}_{ij} = \frac{\hbar^2}{2mc} \left(\partial_i R \partial_j R - R \partial_i \partial_j R \right) . \tag{43}$$

Trace of Energy-Momentum Tensor

$$\mathfrak{T}_i^i = \left\{ 1 + L_C^2 \frac{1 - 3\mu}{4} \Box \right\} (mcR^2) \tag{44}$$

For $\mu = 0$ original Bohm potential.

For $\mu = 1/3$ only classical dust contributes to trace.

For $\mu = 1$ the original Klein-Gordon case.

Energy-Momentum Tensor in scaling variables

Use $R = e^{\sigma}/\sqrt{V}$, then

$$\mathfrak{T}^i_j = \frac{mc}{V} e^{2\sigma} u^i u_j + \hbar^2 \mathfrak{W}^i_j \tag{45}$$

with

$$\mathfrak{W}^i_j = \frac{1}{2mcV} e^{2\sigma} \left[2\mu\partial^i\sigma\,\partial_j\sigma + (\mu-1)\partial^i\partial_j\sigma - \mu\delta^i_j\left(2\partial_k\sigma\partial^k\sigma + \Box\sigma\right)\right] \tag{46}$$

Outline

Einstein equation

Consider the downscaled μ-Madelung-Bohm-Takabayashi energy-momentum as a source of gravity:

$$G^i_j - \Lambda\delta^i_j = \frac{8\pi G}{c^3} e^{-2\sigma}\mathfrak{T}^i_j. \tag{47}$$

Use the Schwarzschild length (half-radius) $L_S = \frac{Gm}{c^2}$, to achieve

$$\begin{aligned} G^i_j - \Lambda\delta^i_j &= \frac{8\pi L_S}{V} u^i u_j + \frac{4\pi L_S L_C^2}{V} \mathfrak{Q}^i_j \\ \mathfrak{Q}^i_j &= 2\mu\partial^i\sigma\partial_j\sigma + (\mu-1)\partial^i\partial_j\sigma - \mu\delta^i_j(2\partial_k\sigma\partial^k\sigma + \Box\sigma). \end{aligned}$$

Conformal transformation by $\Omega(x)$

From (originally flat) spacetime to a curved one:

$$g'_{ik} = e^{2s}\,\eta_{ik} \tag{48}$$

(induced additive change in) Christoffel symbol

$$\delta\Gamma^{j}_{ik} = \partial_i s\,\delta^j_k + \partial_k s\,\delta^j_i - \partial^j s\,\eta_{ik} \tag{49}$$

(induced additive change in) Ricci tensor

$$\delta R_{ik} = 2\,(\partial_i s\,\partial_k s - \partial_i\partial_k s) - \left(\Box s + 2\,\partial_j s\,\partial^j s\right)\,\eta_{ik} \tag{50}$$

(induced additive change in) Ricci scalar

$$\delta\mathcal{R} = -6\,e^{-2s}\,\left(\Box s + \partial_j s\,\partial^j s\right) \tag{51}$$

Transformed Einstein equation

Regard a conformally transformed Einstein tensor.

$$\overline{G}^i_j - \Lambda \delta^i_j = \frac{8\pi G}{c^3}\, e^{-2\sigma}\, \mathfrak{T}^i_j \tag{52}$$

after conformal transformation becomes:

$$\left[G^i_j + \left(2\Box s + \partial_k s \partial^k s - \Lambda \right) \delta^i_j - 2\partial^i \partial_j s + 2\partial^i s\, \partial_j s \right] = \frac{8\pi G}{c^3}\, e^{-2\sigma}\, \mathfrak{T}^i_j. \tag{53}$$

We want to connect s and σ. As a source we insert our μ-Madelung Energy-Momentum-Tensor as obtained above.

Term by term identification

$$G^i_j = \frac{8\pi L_S}{V} u^i u_j$$

$$\left(2\Box s + \partial_k s \partial^k s - \Lambda\right) = -\mu \frac{4\pi L_S L_C^2}{V} \left(2\partial_k \sigma \partial^k \sigma + \Box \sigma\right)$$

$$-2\partial^i \partial_j s = (\mu - 1)\frac{4\pi L_S L_C^2}{V} \partial^i \partial_j \sigma$$

$$2\partial^i s \partial_j s = 2\mu \frac{4\pi L_S L_C^2}{V} \partial^i \sigma \partial_j \sigma$$

Obviously: $s = \sigma$!

We have **classical** dust

$$G^i_j = \frac{8\pi L_S}{V} u^i u_j, \tag{54}$$

we have $\mu = 1/3$, i.e. **conformal** quantum part, and

$$V = \frac{4\pi}{3} L_S L_C^2 \tag{55}$$

is a scaled **Planck volume** (by M_P/m). Finally we have a **cosmolgical** term:

$$\Lambda = 3\left(\Box\sigma + \partial_k\sigma\partial^k\sigma\right) = 3\frac{\Box R}{R} \tag{56}$$

Outline

The Constant Part of the Cosmological Term

For a mass m particle - bound in $-\alpha/r$ potential

$$\langle\Lambda\rangle = -3\,\langle\frac{\nabla^2 R}{R}\rangle = \frac{3}{a^2} \tag{57}$$

with $a = L_C/\alpha = \hbar/\alpha mc$ being the Bohr radius.

An "onium" of two equal masses of m has double Bohr radius (reduced mass is half).

It is nice to use units with $c = 1$. Then

$$\hbar = L_P \cdot M_P; \qquad G = \frac{L_P}{M_P}.$$

Grav-Onium Scenario: formulas

$c = 1$

Gravitational α from

thanks to Antal Jakovác for (re-)forcing this view!

$$\boxed{\frac{\alpha}{r} = \frac{Gm^2}{\hbar\, r} \qquad \Longrightarrow \qquad \alpha = \frac{m^2}{M_P^2}.}$$

Gravi-Bohr radius: $a_B = \dfrac{\hbar}{\alpha m} = L_P \left(\dfrac{M_P}{m}\right)^3.$

Grav-Onium radius: $a = 2a_B$, reduced mass: $m_{\rm red} = m/2.$

Cosmo constant: $L_P^2 \cdot \Lambda = 3L_P^2/a^2 = 0.75\,(m/M_P)^6.$

Grav-Onium Scenario: numbers

$c = 1$

Planck scale: $L_P \approx 1.6 \cdot 10^{-20}$ fm $\quad M_P \approx 1.2 \cdot 10^{19}$ GeV.

Cosmo constant: $\Lambda \approx 10^{-82}$ fm^{-2}

in natural units: $L_P^2 \cdot \Lambda \approx 2.56 \cdot 10^{-122}$

The reduced mass ratio: $m/M_P \approx 5.7 \cdot 10^{-21}$.

The grav-onium mass: $\mathbf{2m \approx 137}$ **MeV.**

Summary

- QM is off-mass-shell ($\hbar^2 \Box R / R$) for free particles.
- Via conformal trf a cosmological term ($\Lambda = 3 \Box R / R$) appears.
- Natural reference volume: $V = \frac{4\pi}{3} L_S L_C^2$.
- $\mu = 1/3$: $\boxed{\text{classical trace}}$ and $\boxed{\text{quantum dilaton}}$.
- Grav-Onium mass is around the π^0-mass from cosmological constant

BACKUP SLIDES

Outlook

- Outlook
  - Delphenic: Madelung ansatz for Pauli and Dirac.
  - Delphenic: Madelung as a conformal transformation
  - Jackiw: nonabelian external fields.
  - Brans-Dicke etc.: Dilaton type actions.

Madelung fluid density

Canonical momenta from $\mathfrak{S} = \int \mathfrak{L}\, d^3x dt$:

$$\Pi_R = \frac{\partial \mathfrak{L}}{\partial \nabla R} = \frac{\hbar^2}{m} \nabla R, \qquad \Pi_\alpha = \frac{\partial \mathfrak{L}}{\partial \nabla \alpha} = \frac{R^2}{m} \nabla \alpha,$$

$$P_R = \frac{\partial \mathfrak{L}}{\partial \frac{\partial R}{\partial t}} = 0, \qquad P_\alpha = \frac{\partial \mathfrak{L}}{\partial \frac{\partial \alpha}{\partial t}} = R^2 \tag{58}$$

Continuity eq:

$$\frac{\partial P_\alpha}{\partial t} + \nabla \Pi_\alpha = 0$$

fluid density $\boxed{\rho = P_\alpha = R^2}$.

Madelung current

The "classical" momentum defines a velocity as $\vec{P} = m\vec{v}$

The continuity equation reads as

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0. \tag{59}$$

Bohm potential

The quantum correction to the energy can be expressed as

$$-\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} = -\frac{\hbar^2}{2m}\left(\frac{\nabla^2 \rho}{2\rho} - \frac{(\nabla \rho)^2}{4\rho^2}\right) \tag{60}$$

Fisher entropy

The action expressed by ρ (up to surface terms) becomes

$$\mathfrak{S} = \int \left[\boxed{\left(\frac{P^2}{2m} + V - E \right) \rho} + \boxed{\left(\frac{\hbar^2}{2m} \frac{(\nabla \rho)^2}{4\rho} \right)} \right] d^3x \, dt \qquad (61)$$

The last term, the quantum part, looks like *Fisher entropy*.

Fisher entropy may also have to do with uncertainity (in phase space, however).

Interlude: Dilaton

The improved Energy-Momentum-Tensor (arxiv:0307199)

- Canonical EMT: infinitesimal shift in x^μ generates it
- Neither symmetric nor gauge invariant
- Not traceless even for scale-invariant $\mathcal{L}$.

Cure by adding a term with vanishing divergence

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\kappa f^{\kappa\mu\nu}$$

anti-symmetric in κ, ν.

Inf Sym Trf

Canonical Momentum: $$\Pi_i^\mu := \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi^i}.$$

Transformation: $$x' = \Omega x, \quad \varphi'(x') = U\varphi\left(\Omega^{-1} x'\right)$$

Infinitesimal: $$\delta x^\mu = \epsilon_a G^{a\mu}, \quad \delta\varphi^i = \epsilon_a \left(F^{ai} - G^{a\mu}\partial_\mu \varphi^i\right)$$

Conserved Current: $$\epsilon_a \cdot J^{a\mu} = \Pi_i^\mu \delta\varphi^i - \Theta^\mu_\nu \delta x^\nu$$

Antisymetric *correction* to the Noether current is possible.

Dilatation

$$\delta\varphi^i = d \cdot \epsilon\,\varphi^i; \qquad \delta x^\nu = \epsilon\, x^\nu.$$

Conserved dilatation Noether current:

$$J^\mu = x^\nu \Theta^\mu_\nu - d \cdot \varphi^i \Pi^\mu_i.$$

The zero divergence criterion (using that of $\Theta_{\mu\nu}$):

$$\partial_\mu J^\mu = \Theta^\mu_\mu - d \cdot \partial_\mu \left(\varphi^i \Pi^\mu_i\right) = 0.$$

So in order to have scale invariance *we add* a divergenceless part to achieve zero trace of $T_{\mu\nu}$!

Our Case

For the complex KG Lagrangian:

$$\varphi^i \Pi_i^\mu = \frac{1}{2}\left(\psi \partial^\mu \psi^* + \psi^* \partial^\mu \psi\right) = \frac{\hbar^2}{mc}\,\partial^\mu\, R^2 = L_C^2\,\partial^\mu(mcR^2).$$

Correction to trace of EMT:

$$\mathfrak{T}^\mu_{\ \mu} = (\partial_\mu J^\mu)_{\mathrm{non-scaling\ \ part}} + d \cdot L_C^2\,\Box(mcR^2).$$

Conclusion: $\boxed{d \,=\, (1-3\mu)/4 \,=\, 0}$.