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Statistical Power Law due to Reservoir Fluctuations

and the Universal Thermostat Independence Principle

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Finite Reservoirs

- Avogadro number (atoms in classical matter) $\sim 6 \cdot 10^{23}$
- Neurons in human brain $\sim 10^{11}$
- New particles from heavy ion collisons $\sim 600 - 6000$
- From elementary high energy collisions (pp) $\sim 6 - 60$

General expectation:

smaller size $\rightarrow$ larger *relative* fluctuations.

Outline

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Ideal Gas: microcanonical statistical weight

The one-particle energy, ω, out of total energy, E, is distributed in a one-dimensional relativistic jet according to a statistical weight factor which depends on the number of particles in the reservoir, n:

$$P_1(\omega) = \frac{\Omega_1(\omega)\,\Omega_n(E-\omega)}{\Omega_{n+1}(E)} = \rho(\omega) \cdot \frac{(E-\omega)^n}{E^n} \qquad (1)$$

Superstatistics: n itself has a distribution (based on the physical model of the reservoir and on the event by event detection of the spectra).

Ideal Reservoir: (Negative) binomial n-distribution

n particles among k cells: bosons $\binom{n+k}{n}$ fermions $\binom{k}{n}$

A subspace (n, k) out of (N, K)

Limit: $K \to \infty$, $N \to \infty$; average occupancy $f = N/K$ is fixed.

$$B_{n,k}(f) := \lim_{K\to\infty} \frac{\binom{n+k}{n}\binom{N-n+K-k}{N-n}}{\binom{N+K+1}{N}} = \binom{n+k}{n} f^n (1+f)^{-n-k-1}. \tag{2}$$

$$F_{n,k}(f) := \lim_{K\to\infty} \frac{\binom{k}{n}\binom{K-k}{N-n}}{\binom{K}{N}} = \binom{k}{n} f^n (1-f)^{k-n}. \tag{3}$$

Bosonic reservoir

Reservoir in hep: E is fixed, n fluctuates, e.g. according to NBD.

$$\sum_{n=0}^{\infty} \left(1 - \frac{\omega}{E}\right)^n B_{n,k}(f) = \left(1 + f\frac{\omega}{E}\right)^{-k-1} \tag{4}$$

Note that $\langle n \rangle = (k+1)f$ for NBD. Then with $T = E/\langle n \rangle$ and $q - 1 = \frac{1}{k+1}$ we get

$$\left(1 + (q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}}$$

This is **exactly** a $q > 1$ Tsallis-Pareto distribution.

Fermionic reservoir

E is fixed, n is distributed according to BD:

$$\sum_{n=0}^{\infty} \left(1 - \frac{\omega}{E}\right)^n F_{n,k}(f) = \left(1 - f\frac{\omega}{E}\right)^k \tag{5}$$

Note that $\langle n \rangle = kf$ for BD. Then with $T = E/\langle n \rangle$ and $q - 1 = -\frac{1}{k}$ we get

$$\left(1 + (q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}}$$

This is **exactly** a $q < 1$ Tsallis-Pareto distribution.

Boltzmann limit

In the $k \gg n$ limit (low occupancy in phase space)

$$\binom{n+k}{n} f^n (1+f)^{-n-k-1} \longrightarrow \frac{k^n}{n!}\left(\frac{f}{1+f}\right)^n \ldots$$
$$\binom{k}{n} f^n (1-f)^{k-n} \longrightarrow \frac{k^n}{n!}\left(\frac{f}{1-f}\right)^n \ldots \qquad (6)$$

After normalization this is the **Poisson** distribution:

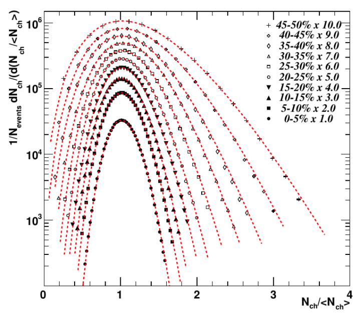
$$\Pi_n = \frac{\langle n \rangle^n}{n!} e^{-\langle n \rangle} \quad \text{with} \quad \langle n \rangle = k \frac{f}{1 \pm f} \qquad (7)$$

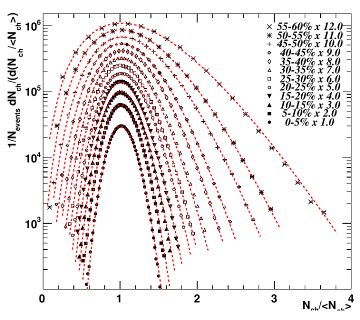
The result is **exactly** the Boltzmann-Gibbs weight factor:

$$\sum_{n=0}^{\infty} \left(1 - \frac{\omega}{E}\right)^n \Pi_n(\langle n \rangle) = e^{-\omega/T}. \qquad (8)$$

Experimental NBD distributions PHENIX PRC 78 (2008) 044902

Au + Au collisons at $\sqrt{s_{NN}} = 62$ (left) and 200 GeV (right). Total charged multiplicities.





$$k \approx 10 - 20 \quad \to \quad q \approx 1.05 - 1.10.$$

Summary of ideal reservoir fluctuations

In all the three above cases

$$T = \frac{E}{\langle n \rangle}, \quad \text{and} \quad q = \frac{\langle n(n-1) \rangle}{\langle n \rangle^2} \tag{9}$$

Ideal gas with general n-fluctuations

Canonical approach: expansion for small $\omega \ll E$.
Tsallis-Pareto distribution as an approximation:

$$\left(1+(q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}} = 1 - \frac{\omega}{T} + q\frac{\omega^2}{2T^2} - \ldots \qquad (10)$$

Ideal reservoir phase space up to the subleading canonical limit:

$$\left\langle \left(1-\frac{\omega}{E}\right)^n \right\rangle = 1 - \langle n \rangle \frac{\omega}{E} + \langle n(n-1) \rangle \frac{\omega^2}{2E^2} - \ldots \qquad (11)$$

To subleading in $\omega \ll E$

$$\mathbf{T} = \frac{\mathbf{E}}{\langle \mathbf{n} \rangle}, \qquad \mathbf{q} = \frac{\langle \mathbf{n}(\mathbf{n}-1) \rangle}{\langle \mathbf{n} \rangle^2} = 1 - \frac{1}{\langle n \rangle} + \frac{\Delta n^2}{\langle n \rangle^2}. \qquad (12)$$

General system with general reservoir fluctuations

Canonical approach: expansion for small $\omega \ll E$.

$$\left\langle \frac{\Omega_n(E-\omega)}{\Omega_n(E)} \right\rangle = \left\langle e^{S(E-\omega)-S(E)} \right\rangle = \left\langle e^{-\omega S'(E)+\omega^2 S''(E)/2-\ldots} \right\rangle$$

$$= 1 - \omega \langle S'(E) \rangle + \frac{\omega^2}{2} \left\langle S'(E)^2 + S''(E) \right\rangle - \ldots \tag{13}$$

Compare with expansion of Tsallis

$$\left(1 + (q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}} = 1 - \frac{\omega}{T} + q\frac{\omega^2}{2T^2} - \ldots \tag{14}$$

Interpret the parameters

$$\frac{1}{T} = \langle S'(E) \rangle, \qquad q = 1 - \frac{1}{C} + \frac{\Delta T^2}{T^2} \tag{15}$$

$\langle S''(E) \rangle = -1/CT^2$ expressed via the heat capacity of the reservoir, $1/C = dT/dE$

Understanding the parameter q

Opposite sign contributions from $\langle S'^2 \rangle - \langle S' \rangle^2$ and from $\langle S'' \rangle$.

In all cases approximately

$$q = 1 - \frac{1}{C} + \frac{\Delta T^2}{T^2}.$$

- $q > 1$ and $q < 1$ are both possible
- for any relative variance $\Delta T / T = 1/\sqrt{C}$ it is exactly $q = 1$
- for $nT = E/\text{dim} = \text{const}$ it is $\Delta T / \langle T \rangle = \Delta n / \langle n \rangle$.
- for ideal gas and n distributed as NBD or BD, the Tsallis form is exact

$K(S)$ additive S non-additive

Use $K(S)$ instead of S to gain more flexibility for handling the subleading term in ω!

$$\left\langle e^{K(S(E-\omega))-K(S(E))} \right\rangle = 1 - \omega \left\langle \frac{d}{dE} K(S(E)) \right\rangle + \frac{\omega^2}{2} \left\langle \frac{d^2}{dE^2} K(S(E)) + \left(\frac{d}{dE} K(S(E)) \right)^2 \right\rangle + \dots \qquad (16)$$

Note that

$$\frac{d}{dE} K(S(E)) = K' S', \qquad \frac{d^2}{dE^2} K(S(E)) = K'' S'^2 + K' S'' \qquad (17)$$

Compare this with the Tsallis power-law!

Tsallis parameters for $K(S)$ entropy

Using previous average notations and assuming that $K(S)$ is independent of the reservoir fluctuations (*universality*):

$$
\begin{aligned}
\frac{1}{T_K} &= K' \frac{1}{T}, \\
\frac{q_K}{T_K^2} &= \left(K'' + K'^2\right) \frac{1}{T^2} \left(1 + \frac{\Delta T^2}{T^2}\right) - K' \frac{1}{CT^2}.
\end{aligned} \tag{18}
$$

By choosing a particular $K(S)$ we can manipulate q_k.

q_K parameter for $K(S)$ entropy

Introduce $F = 1/K' = T_K/T$ and $\Delta T^2/T^2 = \lambda/C$.

Then

$$q_K = \left(1 + \frac{\lambda}{C}\right) (1 - F') - \frac{1}{C} F \tag{19}$$

delivers the simple diff.eq.

$$(\lambda + C)F' + F = \lambda + C(1 - q_K) = 1 + C(q - q_k). \tag{20}$$

With $F(0) = 1$ the only solution for $q_K = q$ is $F = 1, K(S) = S$.

Purposeful deformation achieves $q_K = 1$

Demanding $q_K = 1$ (K-additivity),
Eq.(20) becomes easily solvable with $F(0) = 1/K'(0) = 1$.

Additivity Restoration Condition - ARC ☺

$$(\lambda + \boldsymbol{C})\ \boldsymbol{F}' + \boldsymbol{F} = \lambda \qquad (21)$$

Best handling of subleading terms: UTI principle

Not considering reservoir fluctuations: $\Delta T/T = 0.$

Applying our previous general result for $\lambda = 0$ we obtain

$$F' + \frac{1}{C} F = 0. \tag{22}$$

By this, one arrives at the original Universal Thermostat Independence (UTI) diff. equation:

$$\frac{K''}{K'} = \frac{1}{C}. \tag{23}$$

Deformed entropy formula

T.S.Biró, P.Ván, G.G.Barnaföldi, EPJA 49: 110, 2013

For ideal gas C is constant, without reservoir fluctuations $q = 1 - 1/C$ and $C = 1/(1-q)$.

The solution of eq.(23) with $K(0) = 0,\ K'(0) = 1$ delivers

$$K(S) = C\left(e^{S/C} - 1\right) \tag{24}$$

and one arrives upon using $K(S) = \sum_i p_i K(-\ln p_i)$ at the statistical entropy formulas of Tsallis and Rényi:

$$K(S) = \frac{1}{1-q}\sum_i \left(p_i^q - p_i\right), \qquad S = \frac{1}{1-q}\ln\sum_i p_i^q \tag{25}$$

Deformed formula with C and λ constant

Demanding $q_K = 1$, due to $F = 1/K'$ one obtains in general the diff.eq.

$$\lambda K'^2 - K' + (C + \lambda)\, K'' = 0. \tag{26}$$

First integral (with constant λ and $C_\Delta = \lambda + C$)

$$K'(S) = \frac{1}{(1-\lambda)e^{-S/C_\Delta} + \lambda} \tag{27}$$

Second integral of the DE ☺

$$K(S) = \frac{C_\Delta}{\lambda} \ln\left(1 - \lambda + \lambda e^{S/C_\Delta}\right). \tag{28}$$

$K(S)$-additive composition rule

With the result (28) the $K(S)$-additive composition rule, $K(S_{12}) = K(S_1) + K(S_2)$, is equivalent to

$$h(S_{12}) = h(S_1) + h(S_2) + \frac{\lambda}{C_\Delta} h(S_1) h(S_2) \tag{29}$$

with

$$h(S) = C_\Delta \left(e^{S/C_\Delta} - 1 \right). \tag{30}$$

This is a combination of the ideal gas entropy-deformation, $h(S)$ and an Abe – Tsallis composition law with $q - 1 = \lambda / C_\Delta$.

Non-extensive limit?

Using the auxiliary $h_C(S) = C(e^{S/C} - 1)$ function,

our result is ☺

$$K_\lambda(S) = h^{-1}_{C_\Delta/\lambda}\left(h_{C_\Delta}(S)\right).$$

For $\lambda = 1$ it is obviously $K_1(S) = S$. For $\lambda = 0$ we get $K_0(S) = h_C(S)$.

A particular limit: $C \to \infty, \lambda \to \infty$ but $\lambda / C_\Delta \to \tilde{q} - 1$ finite:

$$K_{NE}(S) = h^{-1}_{1/(\tilde{q}-1)}(S) = \frac{1}{\tilde{q}-1} \ln\left(1 + (\tilde{q}-1)S\right). \qquad (31)$$

K-additivity: $S_{12} = S_1 + S_2 + (\tilde{q} - 1) S_1 S_2$.

Generalized Tsallis formula

based on $K(S) = \sum_i p_i K(-\ln p_i)$ ☺

$$K_\lambda(S) = \frac{C_\Delta}{\lambda} \sum_i p_i \ln\left(1 - \lambda + \lambda p_i^{-1/C_\Delta}\right). \tag{32}$$

Normal fluctuations: $K_1(S) = -\sum_i p_i \ln p_i$ is exactly the Boltzmann entropy!

No fluctuations: $K_0(S) = C\sum_i \left(p_i^{1-1/C} - p_i\right)$ is Tsallis entropy with $q = 1 - 1/C$.

Extreme large fluctuations and arbitrary $C(S)$:

$$K_\infty(S) = \ln(1+S) = \sum_i p_i \ln(1 - \ln p_i). \tag{33}$$

The canonical p_i distribution is Lambert W, it shows tails like the Gompertz distribution

Outline

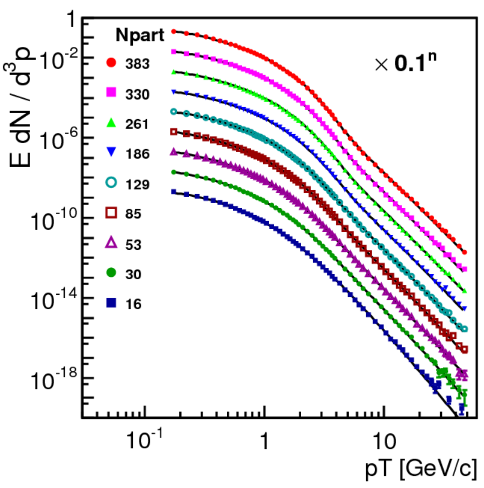
Statistical vs QCD power-law

The experimental fact for hadrons is **NBD**!

- QCD power-law: constant power $(k+1) > 4$ (conformal limit)
- statistical power: $(k+1) = \langle n \rangle / f \propto$ reservoir size
- data fits: ALICE LHC $k+1$ powers vs N_{part}
- soft and hard power-laws differ for large N_{part}

Soft and Hard Tsallis fits:

ALICE PLB 720 (2013) 52



change at $p_T = 4$ GeV.

Trends with N_{part}

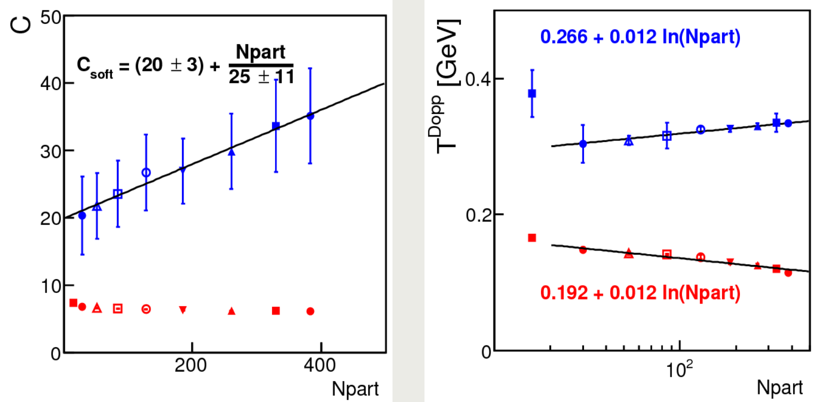


Figure : $C = k + 1$ powers of the power law and fitted T parameters.

Summary

- There are $S'(E)$-temperature fluctuations due to finite reservoirs; they are never *exactly* Gaussian.
- Ideal gas reservoirs with NBD or BD number fluctuations lead to exact Tsallis distributions: $q = 1 + \frac{1}{k+1}$ and $q = 1 - \frac{1}{k}$.
- Tsallis distribution is the approximate canonical weight with fluctuating reservoirs: $q = 1 - 1/C + \Delta T^2/T^2$.
- A general method (UTI) is presented to derive optimally additive entropy: $K(S)$ for $q_K = 1$.
- New entropy formula; for infinite temperature fluctuations at finite heat capacity it is *parameter – free.*

$$K(S) = \sum_i p_i \ln\left(1 - \ln p_i\right) .$$