Thermodynamics, plasticity and rheology

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– Minimal, simple small strain viscoelastoplasticity
  – Minimal – reduction of empirical freedom
    plasticity, rheology
  – Simple – differential equation (1D, homogeneous)

Min?

Classical plasticity

1. Elastic stress strain relations
2. Plastic strain: $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$
3. Yield condition – *f*: $f(\sigma,...) = 0$

   $(\dot{f}(\sigma,...) = 0)$
4. Plastic potential – *g*: $\dot{\boldsymbol{\varepsilon}}^p = \lambda \dfrac{\partial g}{\partial \sigma}(\sigma,...)$

   (non-associative)
5. Evolution of plastic strain

   (non-ideal)

→ Three empirical functions:

static potential *s*, yield function *f*, plastic potential *g*

→ Second Law?

Min?

Thermodynamic plasticity (Ziegler, Maugin, …, Bazant, Houlsby,…)

1. Thermostatics – elastic stress strain relations
2. Plastic strain is a special internal variable
3. Dissipation potential = plastic potential

$$\left(Td_iS=\right)\Phi(\dot{\boldsymbol{\varepsilon}}^p,\ldots)=\mathbf{F}:\dot{\boldsymbol{\varepsilon}}^p$$

4. First order homogeneity

$$F=\nu\frac{\partial\Phi}{\partial\dot{\boldsymbol{\varepsilon}}^p}$$

→ Relation to Second Law is established, evolution of plastic strain is derived.

→ Two empirical functions:

static potential s, dissipation potential Φ

Min?

Thermodynamic rheology (Verhás, 1997)

$$s(e,\mathbf{F},\boldsymbol{\xi}) = s_0(e,\mathbf{F}) - \frac{1}{2}\boldsymbol{\xi} : \boldsymbol{\xi}$$ entropy

Entropy balance:

$$\rho\dot{s} + \nabla\cdot\mathbf{j}_s = \sigma_s = \mathbf{j}_q\cdot\nabla\frac{1}{T} + \frac{1}{T}\left(\mathbf{T} + \rho T\mathbf{F}\frac{\partial s}{\partial \mathbf{F}}\right) : \left(\dot{\mathbf{F}}\mathbf{F}^{-1}\right) - \frac{\rho}{T}\boldsymbol{\xi} : \dot{\boldsymbol{\xi}} \geq 0,$$

$$\boxed{TP_s = \left(\boldsymbol{\sigma} + \rho\frac{\partial s}{\partial \boldsymbol{\varepsilon}}\right) : \dot{\boldsymbol{\varepsilon}} - \rho\boldsymbol{\xi} : \dot{\boldsymbol{\xi}} \geq 0,}$$ isothermal, small strain

	Force	Flux – material function
Mechanical	$(\nabla \circ \mathbf{v}) = \dot{\boldsymbol{\varepsilon}}$	$\boldsymbol{\sigma} + \rho\frac{\partial s}{\partial \boldsymbol{\varepsilon}}$
Rheological	$-\rho\boldsymbol{\xi}$	$\dot{\boldsymbol{\xi}} = \mathbf{G}(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}, \boldsymbol{\xi})$

Min?

Linear relations:

$$\begin{pmatrix} \boldsymbol{\sigma}^{v} = \boldsymbol{\sigma} + \rho \dfrac{\partial s}{\partial \boldsymbol{\varepsilon}} \\ \dot{\boldsymbol{\xi}} \end{pmatrix} = \mathbf{L} \begin{pmatrix} \dot{\boldsymbol{\varepsilon}} \\ -\rho \boldsymbol{\xi} \end{pmatrix} = \dots = \begin{pmatrix} l_1 & l_{12} \\ l_{21} & l_2 \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{\varepsilon}} \\ -\rho \boldsymbol{\xi} \end{pmatrix},$$

isotropy, one dimension, …

Ideal elastic: $\rho \dfrac{\partial s}{\partial \varepsilon} = \rho \dfrac{\partial}{\partial \varepsilon}(-\mu \varepsilon : \varepsilon) = -2G\varepsilon,$

Second Law: $l_1 > 0, \quad l_2 > 0, \quad l_1 l_2 - l_{12} l_{21} > 0.$

The internal variable can (must?) be eliminated:

$$\boxed{\sigma + \tau \dot{\sigma} = 2\eta \tau_d \ddot{\varepsilon} + 2\eta \dot{\varepsilon} + 2G\varepsilon}$$

Jeffreys body

relaxation material inertia creep

$$\tau = \frac{\rho}{l_2} > 0, \quad \tau_d = \frac{\rho l_1}{2\eta l_2} > 0, \quad 2\eta = \frac{l_1 l_2 - l_{12} l_{21} + 2G\rho}{l_2} > 0.$$

Further motivation:

- Rheology (creep *and* relaxation)+plasticity
- Non associated flow – soils

dissipation potentials = normality

= (nonlinear) Onsagerian symmetry ?

Truesdell + dual variables (JNET, 2008, **33**, 235-254.)

rheological thermodynamics (KgKK, 2008, **6**, 51-92)

Is the thermodynamic framework too strong??

→ computational and numerical stability

min+min=simple?

Unification:

dissipation potential – constitutive relations

$$s(e,\boldsymbol{\varepsilon},\boldsymbol{\xi}) = s_0(e) - \frac{G}{2}(\boldsymbol{\varepsilon}-\boldsymbol{\xi})^2 - \frac{\overline{G}}{2}\boldsymbol{\xi}^2$$

$$\xi \to \boldsymbol{\varepsilon}^p$$

$$\boxed{TP_s = \left(\boldsymbol{\sigma} + \rho\frac{\partial s}{\partial \boldsymbol{\varepsilon}}\right) : \dot{\boldsymbol{\varepsilon}} - \rho\frac{\partial s}{\partial \boldsymbol{\varepsilon}^p} : \dot{\boldsymbol{\varepsilon}}^p \geq 0,}$$

$$\sigma + \rho\frac{\partial s}{\partial \varepsilon} = l_1\,\dot{\varepsilon} + l_{12}\frac{\partial s}{\partial \varepsilon^p},$$

$$\rho\dot{\varepsilon}^p = l_{21}\,\dot{\varepsilon} + l_2\frac{\partial s}{\partial \varepsilon^p},$$

$$\boxed{l_2 = l + \frac{\left|\dot{\boldsymbol{\varepsilon}}^p\right|}{\sigma_{cr}}}$$

Differential equation

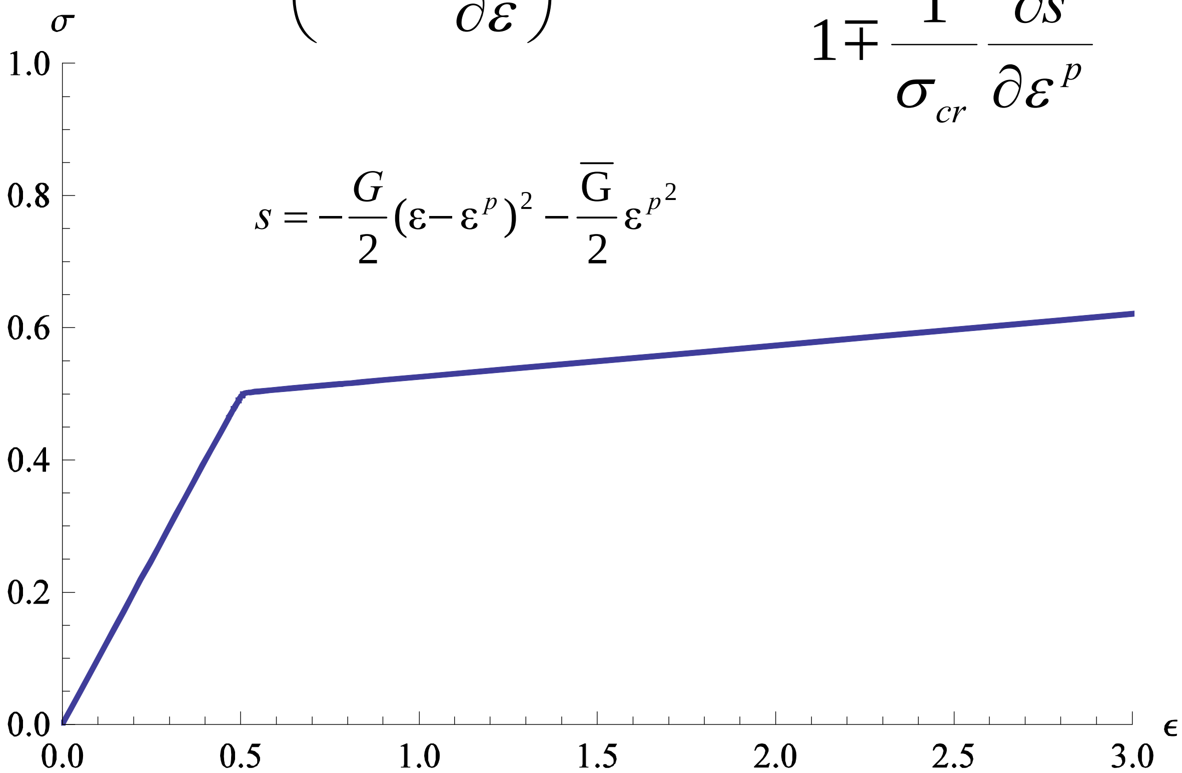
simple

Strain hardening:

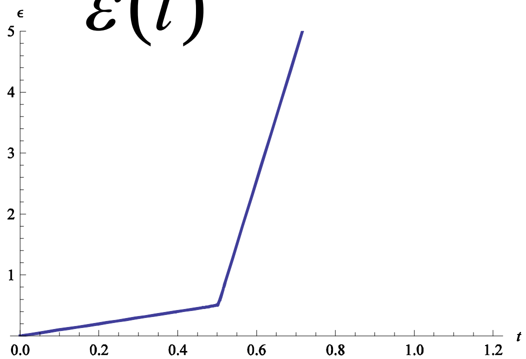
$$G = 1,\ \ \overline{G} = 0.05,\ \ l_1 = 0.001,\ \ l = 0.00005;\ \ \sigma_{cr} = 0.5,\ \ v = 1$$

$$\dot{\varepsilon} = l_1^{-1}\left(\sigma + \rho\frac{\partial s}{\partial \varepsilon}\right), \qquad \rho\dot{\varepsilon}^p = \frac{l\dfrac{\partial s}{\partial \varepsilon^p}}{1 \mp \dfrac{1}{\sigma_{cr}}\dfrac{\partial s}{\partial \varepsilon^p}}$$

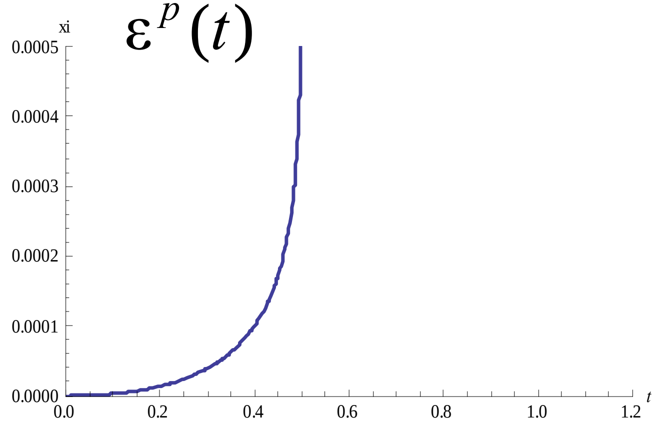
$$s = -\frac{G}{2}(\varepsilon - \varepsilon^p)^2 - \frac{\overline{G}}{2}\varepsilon^{p\,2}$$



$\varepsilon(t)$

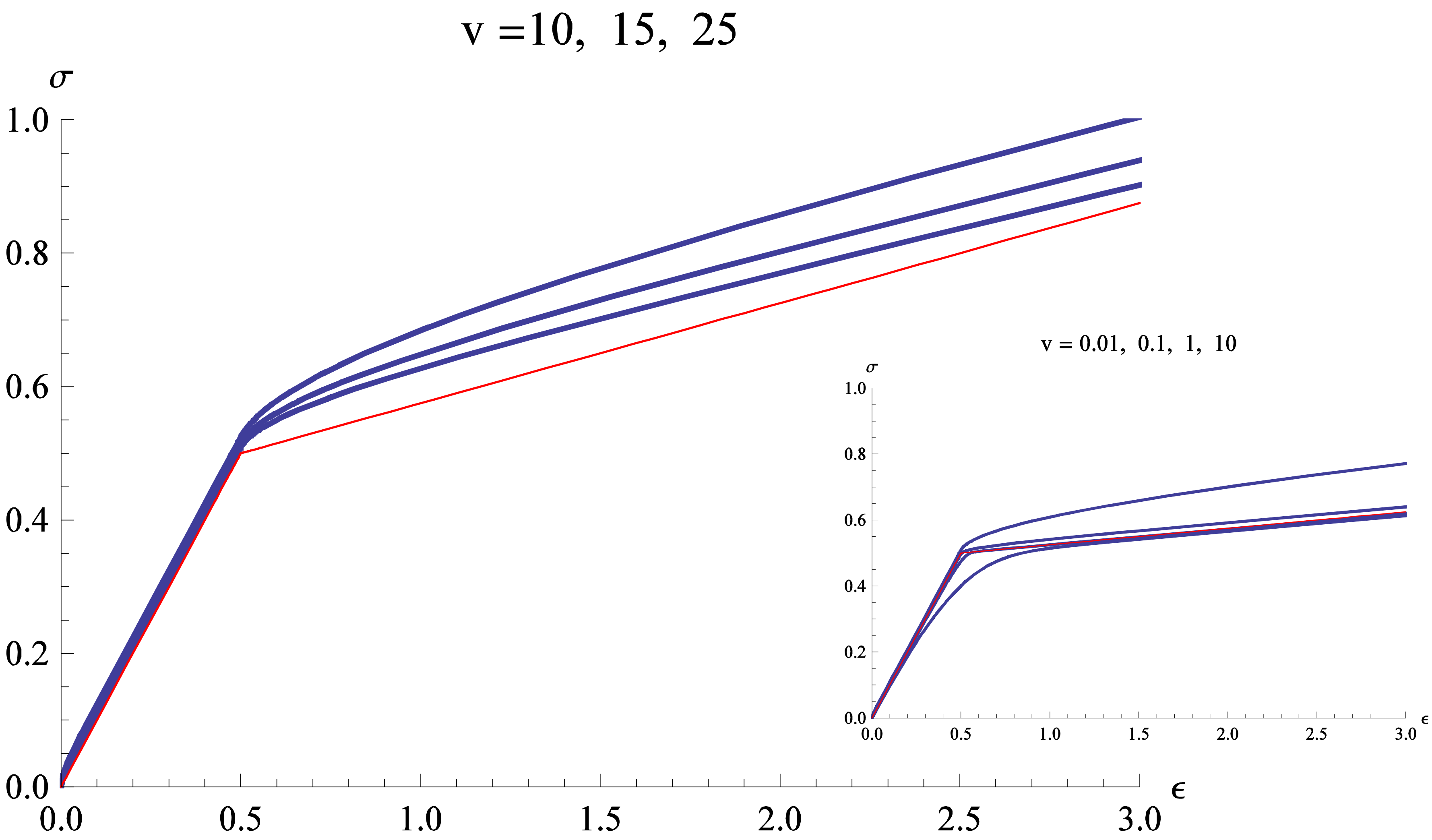


$\varepsilon^p(t)$



simple

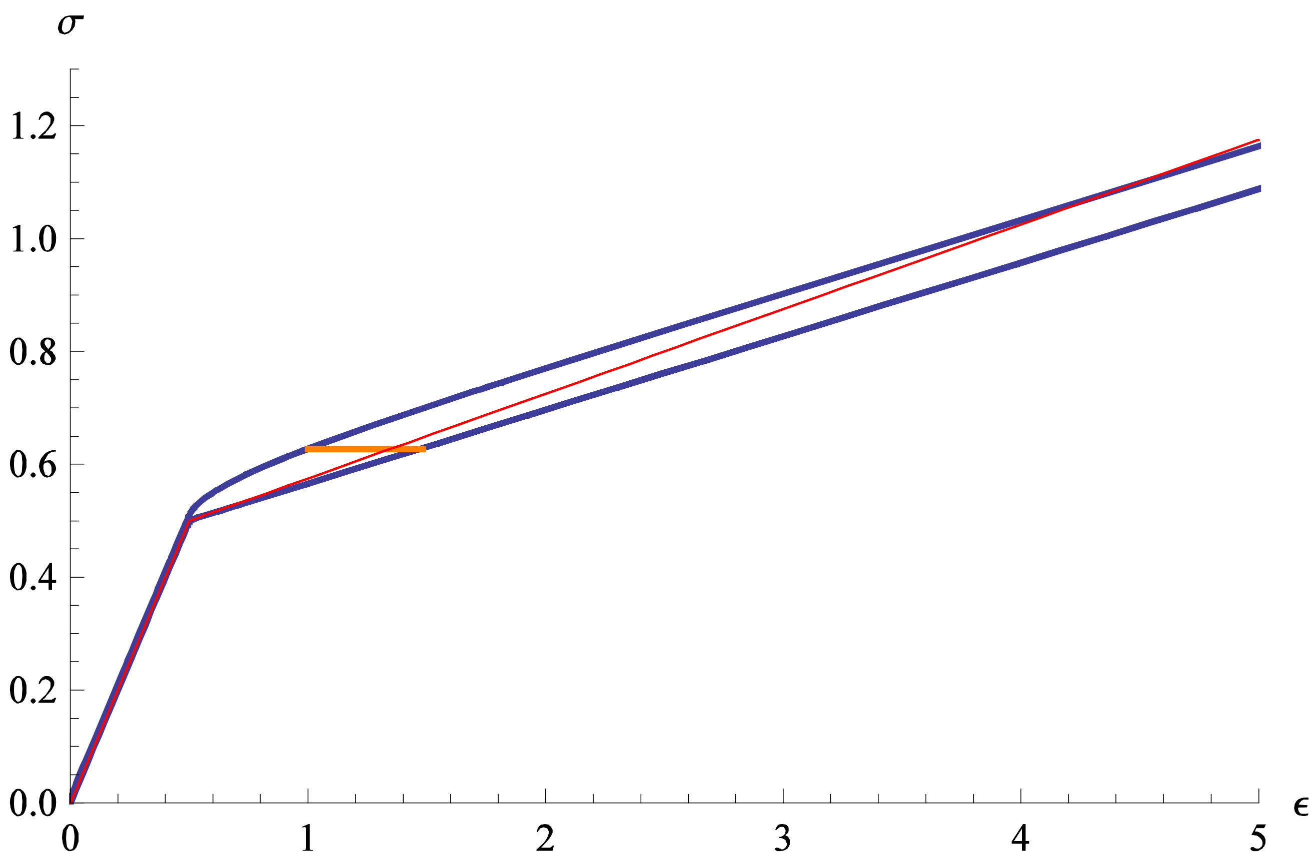
Loading rate dependence:



simple

Creep:

$$G = 1, \ \ \overline{G} = 0.15, \ \ v_1 = 0.1, \ \ v_2 = 10.$$

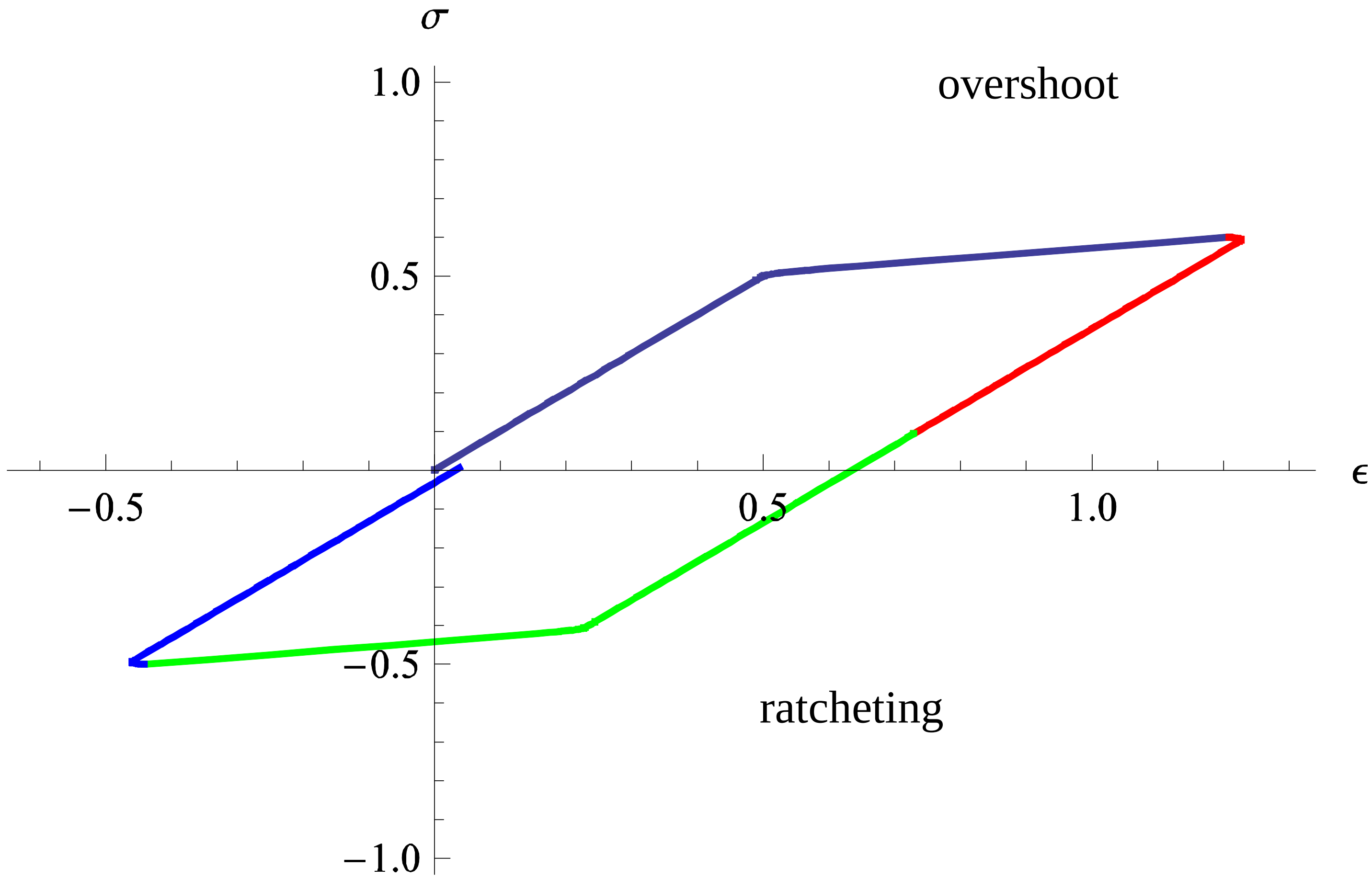


like Armstrong-Frederic kinematic hardening

simple

Hysteresis, Bauschinger effect, ratcheting:

$$G = 1,\ \ \overline{G} = 0.15,\ \ l_1 = 0.001,\ \ l = 0.001;\ \ \sigma_{cr} = 0.5,\ \ \sigma_{Max} = 0.6,\ \ v = 1$$



simple?

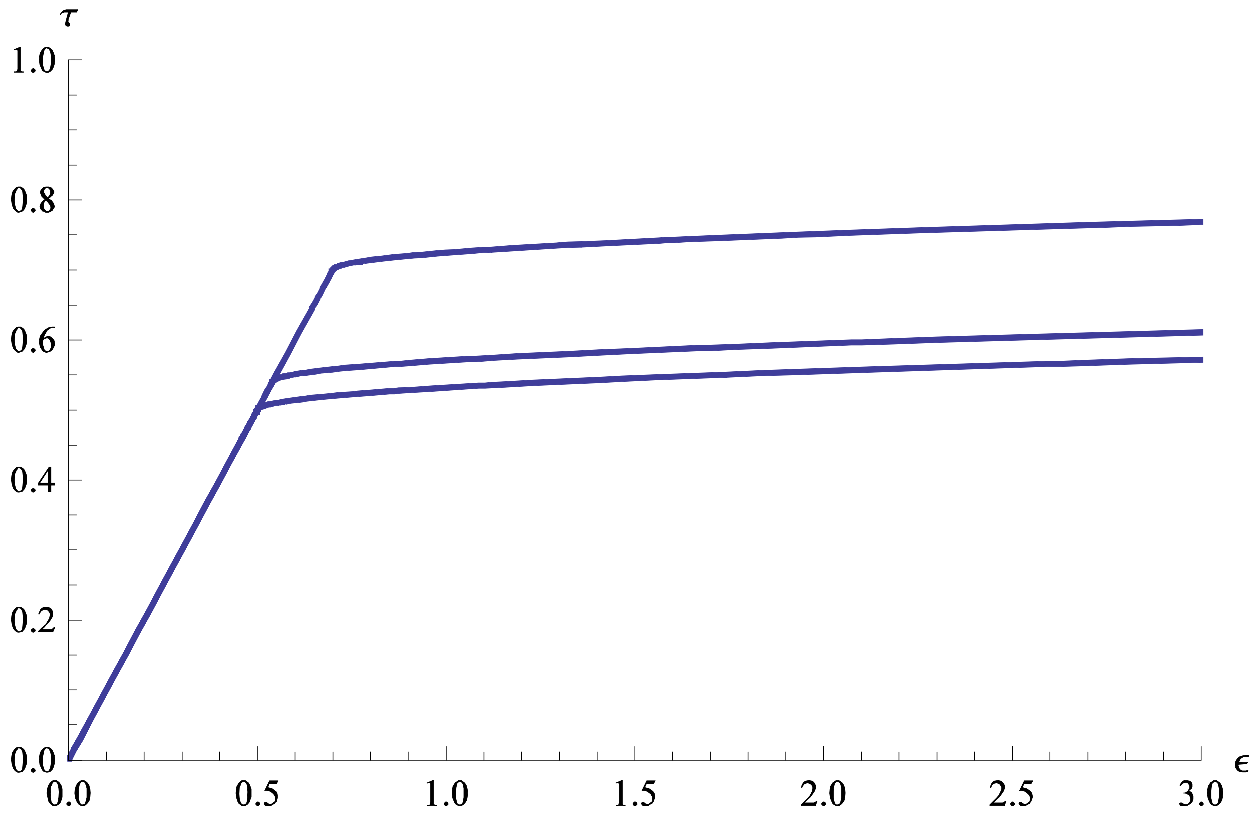
Non-associative, Mohr-Coulomb, dilatancy

(according to Houlsby and Puzrin):

$$s(e,\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p) = s_0(e) - \frac{E}{2}(\varepsilon_1 - b\,\varepsilon_2^p)^2 - \frac{G}{2}(\varepsilon_2 - \varepsilon_2^p)^2 - \frac{\overline{G}}{2}\boldsymbol{\varepsilon}^{p\,2}$$

$$K = 1,\ \ G = 0.5,\ \ \overline{G} = 0,\ \ l_1 = 0.00005,\ \ l = 0.001;\ \ \sigma_{cr} = 0.5,\ \ b = -0.2,\ \ v = 1$$

$\sigma = 0,\ 0.2,\ 1$



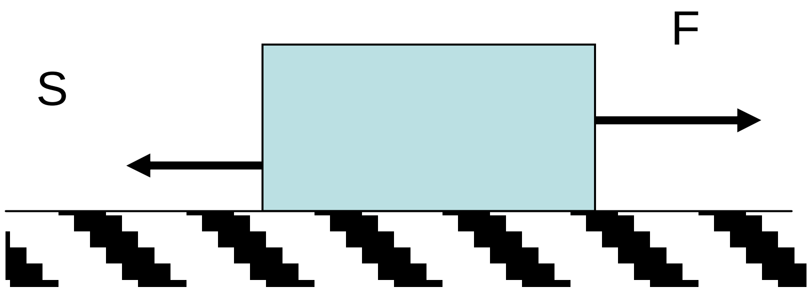
Discussion

1. Minimal viscoelastoplasticity
2. Possible generalizations
   - finite strain,
   - objective derivatives,
   - gradient effects,
   - modified Onsagerian coefficients,
   - yield beyond Tresca, etc...
3. Non-associative? + thermo
4. More general constitutive relations
   (gyroscopic forces?) + thermo
5. Differential equation (can be incremental)
6. Numerical stability (thermo)

Thank you for your attention!

Interlude – damping and friction

Dissipation in contact mechanics:



$$E(x,\dot{x}) = \frac{m\dot{x}^2}{2} + V(x)$$

$$0 \geq -D = \dot{E}(x,\dot{x}) = (m\ddot{x} - F)\dot{x} = S\dot{x}$$

$$S = S_d + S_f = -\beta\dot{x} - \mu N \frac{\dot{x}}{|\dot{x}|} \quad \Rightarrow \quad D = \beta\frac{\dot{x}^2}{2} + \mu N|\dot{x}|$$

Mechanics – constant force $\quad F = \pm\mu N$

Plasticity:

$$S = -\beta\left(1 + \frac{|S|}{\mu N}\right)\dot{x} \;\Rightarrow\; \dot{x} = \frac{-\beta^{-1}S}{1 + \frac{|S|}{\mu N}} \quad \longrightarrow \quad -\mu N \leq S \leq \mu N$$

(spec. $S \parallel \dot{x}$)

Thermodinamics - Mechanics