# The principles of objectivity, relativity and frame indifference in classical physics: a space-time point of view

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# Outline

1. Why we need Galilean relativistic space-time? The irreversible thermodynamics of simple fluids
2. Space and time $\neq$ space-time. Basic concepts of relativity
3. The origin of transformation rules
4. Single component fluids: The balance of mass-momentum-energy density-flux tensor
5. Thermodynamics of motion and entropy production without reference frames

# Once upon a time...: a historical motivation

1. Classical field theories (CFT): Jaumann, Zaremba, Duhem: time derivatives, deformation[1]
   Elastic solids + rheology: body, material manifold, double tensor fields,...
2. Theory of relativity (RT): Galileo, Newton, Mach, Einstein: Galilean transformations
   Special relativity, inertial observers and reference frames, world lines, ...
3. Relativistic fluids: a benchmark.
   Motion, thermodynamics, reference frames -> stability[2]
4. Kinetic theory and Extended Thermodynamics
   No objectivity, Galilean transformations[3]
5. Nonrelativistic: Ryskin, Murdoch, Liu, ..., Muschik and Restuccia, ...
   Relativistic: Carter, de Groot, Sandoval-Villalbazzo and the Mexicans, Kunihiro and the Japanese, Öttinger and GENERIC

[1] Truesdell and Noll, The Non-Linear Field Theories of Mechanics, 1965.
[2] VP, J. of Statistical Mechanics, 02054, 2009, VP and TS Biró, PLB, 709, 106, 2012
[3] Müller and Ruggeri, Rational Extended Thermodynamics, 1998

# Objectivity and reference frame independence

## Material frame indpendence and transformations

- Galilei invariance and transformations
- Rigid body transformations

Transformation rule of Noll (1958):

$$\hat{x}^a = \begin{pmatrix} \hat{t} \\ \hat{x}^i \end{pmatrix} = \begin{pmatrix} t \\ h^i(t) + Q^{ij}(t)x^j \end{pmatrix},$$

where $Q^{-1} = Q^T$ is an orthogonal tensor, $a \in \{0{,}1{,}2{,}3\}$.
Jakobian:

$$\hat{J}^{ab} = \frac{\partial \hat{x}^a}{\partial x^b} = \begin{pmatrix} 1 & 0 \\ \dot{h}^i + \dot{Q}^{ij}x^j & Q^{ij} \end{pmatrix}$$

## Transformation rule:

$$\hat{C}^a = \hat{J}^{ab} C^b$$

# Particular derived transformation rules

$V^i = \dot{h}^i$

## Spatial vectors

$$\begin{pmatrix} 1 & 0 \\ V^i + \dot{Q}^{ij}x^j & Q^{ij} \end{pmatrix} \begin{pmatrix} 0 \\ C^j \end{pmatrix} = \begin{pmatrix} 0 \\ Q^{ij}C^j \end{pmatrix} \quad \rightarrow \quad \hat{C}^i = Q^{ij}C^j.$$

## Galilean transformations ($Q^{ij} = \delta^{ij}$) and four-vectors?

$$\begin{pmatrix} \hat{\rho} \\ \hat{j}^i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ V^i & \delta^{ik} \end{pmatrix} \begin{pmatrix} \rho \\ j^k \end{pmatrix} = \begin{pmatrix} \rho \\ j^i + \rho V^i \end{pmatrix} \quad \rightarrow \quad \begin{matrix} \hat{\rho} = \rho \\ \hat{j}^i = j^i + \rho V^i \end{matrix}$$

Velocity $v^i := \dot{x}^i(t)$. By definition: $\hat{v}^i = \frac{d}{dt}\hat{x}^i = V^i + \dot{Q}^{ij}x^j + Q^{ij}v^j$
This is not three-vector transformation.

## Velocity as four-vector: $\dot{x}^a = (1, v^i)$

$$\begin{pmatrix} \hat{1} \\ \hat{v}^i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ V^i + \dot{Q}^{ij}x^j & Q^{ij} \end{pmatrix} \begin{pmatrix} 1 \\ v^j \end{pmatrix} = \begin{pmatrix} 1 \\ V^i + \dot{Q}^{ij}x^j + Q^{ij}v^j \end{pmatrix}$$

# Balances of simple fluids

**Local**

$$
\begin{aligned}
\partial_t\rho + \partial_k(\rho v^k) &= 0,\\
\partial_t(\rho v^i) + \partial_k(P^{ik} + \rho v^i v^k) &= 0^i,\\
\partial_t e_{tot} + \partial_k(q^k_{tot} + e_{tot} v^k) &= 0.
\end{aligned}
$$

**Substantial**

$$
\begin{aligned}
\dot\rho + \rho\partial_k v^k &= 0,\\
\dot{(\rho v^i)} + \rho v^i\partial_k v^k + \partial_k P^{ik} &= 0^i,\\
\dot e_{tot} + e_{tot}\partial_k v^k + \partial_k q^k_{tot} &= 0.
\end{aligned}
$$

Notation:

- $\partial_t = \frac{\partial}{\partial_t}$, $\partial_i = \nabla$, $v^i = \mathbf{v}$, indices are not coordinates. $i, j, k \in \{1{,}2{,}3\}$
- $e_{tot}$ is the total energy density.

**Transformations**

$v^i$ relative velocity,
$\partial_t + v^i\partial_i = \frac{d}{dt}$, comoving derivative,
$\hat q^i = q^i + e_{tot} v^i$, conductive and convective

# Fluid thermodynamics

**total - kinetic = internal , $e = e_{tot} - \rho v^2/2$**

$$\frac{d}{dt}\left(\rho\frac{v^2}{2}\right) + \rho\frac{v^2}{2}\partial_i v^i + \partial_i(P^{ik}v_k) - P^{ik}\partial_i v_k = 0.$$

$$\dot{e} + e\partial_k v^k + \partial_k(\underbrace{q^k_{tot} - P^{ik}v_i}_{q^k}) + P^{ik}\partial_i v_k = 0.$$

Thermodynamics:

$s(e,\rho)$, $\quad de = Tds + \mu d\rho$; $\quad e + p = Ts + \mu\rho$, $\quad s^i = \frac{q^i}{T}$

$$\dot{s} + s\partial_i v^i + \partial_i s^i = \frac{1}{T}\dot{e} - \frac{\mu}{T}\dot{\rho} + s\partial_i v^i + \partial_i\frac{q^i}{T} =$$

$$-\frac{1}{T}\left(e\partial_i v^i + \partial_i q^i + P^{ij}\partial_i v_j\right) + \frac{\mu}{T}\left(\rho\partial_i v^i\right) + s\partial_i v^i + \frac{\mu}{T}\partial_i q^i + q^i\partial_i\frac{1}{T} =$$

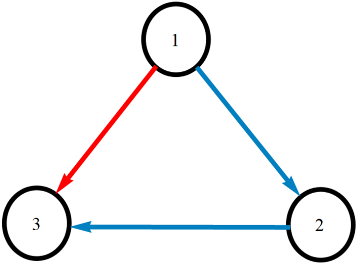
$$\boxed{q^i\partial_i\frac{1}{T} - \frac{1}{T}(P^{ij} - p\delta^{ij})\partial_i v_j \geq 0.}$$

Basic fields: $\rho$, $e$, $v^i$; $\quad$ Constitutive functions: $q^i$, $P^{ij}$

# Kinetic energy and Galiean transformation ?

**Total energy, kinetic energy and internal energy :**

$$e_{tot} = e + \rho\frac{v^2}{2}, \quad \text{therefore} \quad e_2 = e_1 + \rho\frac{v_{12}^2}{2}$$




Transitivity ?

$$e_2 = e_1 + \rho\frac{v_{12}^2}{2}, \quad e_3 = e_2 + \rho\frac{v_{23}^2}{2} \quad \rightarrow \quad e_1 = e_3 + \rho\frac{v_{31}^2}{2}$$

# Problems

## Nonrelativistic

1. Transformation rules. Pressure, heat, energy?
2. $v^i, q^i, P^{ij}, \ldots$ are relative quantities. Is dissipation real/physical/objective?
3. What is moving? Mass? (Frames: Eckart or Landau-Lifsic, Brenner)
4. Local equilibrium? Thermodynamics is comoving with what? Entropy flux?

# Compatibility

## More complex materials

- (Truesdell-)Noll formulation with transformation rules does not help[a],
- Beyond fluids: basic kinematics, many deformation measures, etc ...[b].

[a] Matolcsi-VP, PLA, 353, 109-112, 2006
[b] Fülöp-VP, Mathematical Methods in the Applied Sciences, 35, 1825, 2012

## Relativistic

- What is an ideal fluid? Flows. (Eckart or Landau-Lifsic? Is there a choice? Thermodynamics? Stability? Energy-momentum[a]

[a] VP 2009, VP-Biró, 2012, 2013, arXiv

## Kinetic theory and more

- Moment (or gradient) expansion: series of balances, increasing tensorial order. Transformation rules are inherited[a].

[a] Ruggeri, Continuum Mechanics and Thermodynamics, 1, 3, 1989

What is a fluid? What is moving?

reference frame independent + flow-frame independent =

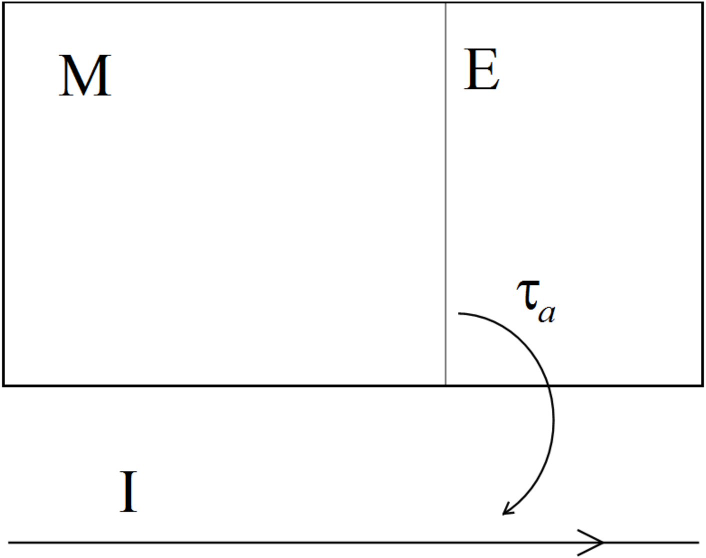
absolute

# Mathematical structure of Galilean relativistic space-time

1. The *space-time* $\mathbb{M}$ is an oriented four dimensional vector space of the $x^a \in \mathbb{M}$ *world points or events*. There are no Euclidean or pseudoeuclidean structures on $\mathbb{M}$: the length of a space-time vector does not exist.
2. The *time* $\mathbb{I}$ is a one dimensional oriented vector space of $t \in \mathbb{I}$ *instants*.
3. $\tau_a : \mathbb{M} \to \mathbb{I}$ is the *timing* or *time evaluation*, a linear surjection.
4. $\delta_{\bar{a}\bar{b}} : \mathbb{E} \times \mathbb{E} \to \mathbb{R} \otimes \mathbb{R}$ Euclidean structure is a symmetric bilinear mapping, where $\mathbb{E} := Ker(\tau) \subset \mathbb{M}$ is the three dimensional vector space of *space vectors*.

- Simplification: space-time and time are affine spaces
- Simplification: measure lines.
- Abstract indexes: $a, b, c, ...$ for $\mathbb{M}$, $\bar{a}, \bar{b}, \bar{c}, ...$ for $S$

# Galilean relativistic space-time



# Vectors and covectors

$V$ is a vector space.
Dual space: $V^* := Lin(V, \mathbb{R})$, covector: $v_a \in V^*$, $v^a v_a \in \mathbb{R}$.
If there is a (pseudo)euclidean form on $V$, then vectors and covectors can be identified: $v_{\bar{a}} \equiv \delta_{\bar{a}\bar{b}} v^{\bar{b}}$.

**Tensors:**

$$Lin(V, U) \equiv U \bigotimes V^* \equiv Bilin(U^* \times V, \mathbb{R})$$

Transpose:
$L^a{}_b = L \in Lin(V, U)$, $(L^a{}_b)^* = L_b{}^a = L^* \in Lin(U^*, V^*)$
$L^a{}_b v^b = v^b {\color{red}L_b{}^a}$ or $u_a L^a{}_b v^b = v^b {\color{red}L_b{}^a} u_a$
Symmetry, trace.

# Cotensor generation

**Derivation**

$f : V \to \mathbb{R}$, $Df : V \to V^* \equiv Lin(V, \mathbb{R})$,
notation: $\partial_a f$
$A : V \to V$, $DA : V \to V \bigotimes V^* \equiv Lin(V, V) \equiv Bilin(V^* \times V, \mathbb{R})$,
notation: $\partial_a A^b$
$B : V \to V^*$, $DB : V \to V^* \bigotimes V^* \equiv Lin(V, V^*) \equiv Bilin(V \times V, \mathbb{R})$,
notation: $\partial_a B_b$

E.g. Vectors are extensives, covector: scalar+vector potential in ED
There is no trace of tensors and cotensors. Mixed tensors do not have symmetric or antisymmetric parts.
Vector fields do not have an external derivative, covector fields do not have divergence.

# Space-time concepts I

### Wordline function

$r^a : \mathbb{I} \to \mathbb{M}$, $\tau_a r^a(t) = t$ smooth, transversal

### Four-velocity

$u^a \in V(1)$, $\tau_a u^a = 1$ Can be a tangent vector of word lines.

Four-velocities do not have a magnitude, angle. The sum of two four-velocities does not exist.

### Relative velocity

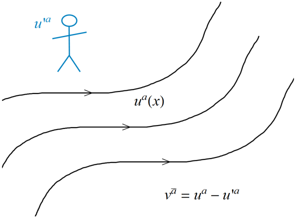
$v^{\bar{a}} = u^a - u'^a$, velocity of $u$ related to $u'$. $\tau_a v^a = 0$.

Spacelike vectors: $\tau_a A^a = 0$. Notation: $A^{\bar{a}}$
Timelike vectors: $\tau_a A^a \neq 0$.

## Reference frames, observers

$u^a : \mathbb{M} \to V(1)$, smooths, global four-velocity field. Tangent vectors of word lines.
Inertial reference frame: $u^a = const.$

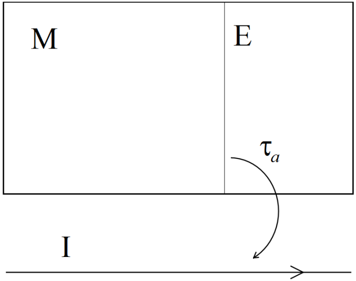
Local space definition by an observer. Space is relative, time is absolute. There is no orthogonality, there is no distinguished projection.
Train space, eagle space and rabbit space.



# Time- and spacelike components: vectors

$$\mathbb{E} \;\underset{\delta^a{}_{\bar b}}{\overset{\pi(u)^{\bar a}{}_b}{\leftrightarrows}}\; \mathbb{M} \;\underset{u^a}{\overset{\tau_a}{\rightleftarrows}}\; \mathbb{I}$$

$$\mathbb{E}^* \;\underset{\pi(u)_b{}^{\bar a}}{\overset{\delta_{\bar b}{}^a}{\leftrightarrows}}\; \mathbb{M}^* \;\underset{\tau_a}{\overset{u^a}{\rightleftarrows}}\; \mathbb{I}^*$$

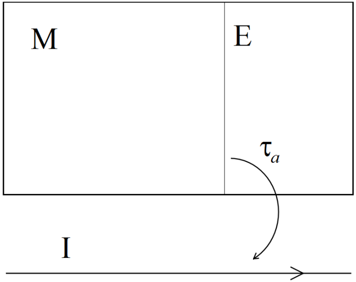



- $\tau_a : \mathbb{M} \to \mathbb{I}$,
- $u^a : \mathbb{I} \to \mathbb{M}$,
- $\pi(u)^{\bar a}{}_b = \delta^a{}_b - u^a \tau_b : \mathbb{M} \to \mathbb{E}$, $u$-projection
- $\delta^a{}_{\bar b} : \mathbb{E} \to \mathbb{M}$, canonical embedding

# Time- and spacelike components: covectors

$$\mathbb{E} \underset{\delta^a_{\bar b}}{\overset{\pi(u)^{\bar a}_{\ b}}{\leftrightarrows}} \mathbb{M} \underset{u^a}{\overset{\tau_a}{\rightleftarrows}} \mathbb{I}$$

$$\mathbb{E}^* \underset{\pi(u)_b^{\ \bar a}}{\overset{\delta_{\bar b}^{\ a}}{\leftrightarrows}} \mathbb{M}^* \underset{\tau_a}{\overset{u^a}{\rightleftarrows}} \mathbb{I}^*$$




- $\tau_a : \mathbb{I}^* \to \mathbb{M}^*, \qquad u^a : \mathbb{M}^* \to \mathbb{I}^*,$
- $\pi(u)_a^{\ \bar b} : \mathbb{E}^* \to \mathbb{M}^*, \qquad \delta_{\bar b}^{\ a} : \mathbb{M}^* \to \mathbb{E}^*.$

$$\tau_a u^a = 1, \quad \tau_a \delta^a_{\bar b} = 0_{\bar b}, \quad \pi(u)^{\bar b}_{\ a} u^a = (\delta^b_{\ a} - u^b \tau_a) u^a = 0^{\bar b}, \quad \pi(u)^{\bar a}_{\ b} \delta^b_{\bar c} = \delta^{\bar a}_{\bar c}.$$

# $u$-form of four-vectors and four-covectors

$$\mathbb{E} \;\overset{\pi(u)^{\bar{a}}{}_{b}}{\underset{\delta^{a}{}_{\bar{b}}}{\leftrightarrows}}\; \mathbb{M} \;\overset{\tau_a}{\underset{u^a}{\rightleftarrows}}\; \mathbb{I}$$

$A^a \in \mathbb{M}$ four-vector
Timelike part: $a := \tau_a A^a$
Spacelike part: $a^{\bar{a}} = \pi^{\bar{a}}{}_{b} A^b = A^a - u^a \tau_b u^b$
$u$-form of a vector: $A^a = a u^a + a^{\bar{a}}$

$$\mathbb{E}^* \;\overset{\delta_{\bar{b}}{}^{a}}{\underset{\pi(u)_{b}{}^{\bar{a}}}{\leftrightarrows}}\; \mathbb{M}^* \;\overset{u^a}{\underset{\tau_a}{\rightleftarrows}}\; \mathbb{I}^*$$

$B_a \in \mathbb{M}^*$ four-covector
Timelike part: $b := u^a B_a$
Spacelike part: $b_{\bar{a}} = \delta_{\bar{a}}{}^{b} B_b$
$u$-form of a covector: $B_a = b\tau_a + \pi_a{}^{\bar{b}} B_{\bar{b}}$

## $u$-form of tensors

### $T^{ab}$ symmetric second order four-tensor

$$T^{ab} = t^a u^b + t^{a\bar{b}} = u^a t^b + t^{\bar{a}b} = t u^a u^b + u^a t^{\bar{b}} + t^{\bar{a}} u^b + t^{\bar{a}\bar{b}},$$

### $R_{ab}$ symmetric second order four-cotensor

$$\begin{aligned} R_{ab} = & \; r_a \tau_b + r_{a\bar{c}} \pi_b^{\bar{c}} = \tau_a r_b + r_{\bar{c}b} \pi_a^{\bar{c}} \\ = & \; r \tau_a \tau_b + r_{\bar{c}} \pi_a^{\bar{c}} \tau_b + r_{\bar{c}} \tau_a \pi_b^{\bar{c}} + r_{\bar{c}\bar{d}} \pi_b^{\bar{c}} \pi_a^{\bar{d}} \\ = & \left( r - 2 r_{\bar{c}} u^c + r_{\bar{c}\bar{d}} u^c u^d \right) \tau_a \tau_b + \left( r_{\bar{b}} - r_{\bar{b}\bar{d}} u^d \right) \tau_a + \left( r_{\bar{a}} - r_{\bar{a}\bar{d}} u^d \right) \tau_b + r_{\bar{a}\bar{b}} \end{aligned}$$

### $Q^a{}_b$ second order mixed four-tensor

$$\begin{aligned} Q^a_b &= q^a \tau_b + \pi_b^{\bar{c}} q^a_{\bar{c}} = q u^a \tau_b + q^{\bar{a}} \tau_b + u^a \pi_b^{\bar{c}} q_{\bar{c}} + q^{\bar{a}}_{\bar{c}} \pi_b^{\bar{c}} \\ &= \left( u^a (q - u^c q_{\bar{c}}) + q^{\bar{a}} - q^{\bar{a}}_{\bar{c}} u^c \right) \tau_b + q_{\bar{b}} u^a + q^{\bar{a}}_{\bar{b}} \end{aligned}$$

## Galilean transformation of four-vectors

$$A^a \overset{u}{\prec} \begin{pmatrix} A \\ A^{\bar{a}} \end{pmatrix}, \qquad A^a \overset{u'}{\prec} \begin{pmatrix} A' \\ A'^{\bar{a}} \end{pmatrix},$$

$$A' = \tau_a A^a = \tau_a (A u^a + A^{\bar{a}}) = A,$$

$$\begin{aligned} A'^{\bar{a}} &= \pi'^{\bar{a}}{}_b A^b = \left(\delta^a{}_b - u'^a \tau_b\right)\left(A u^b + A^{\bar{b}}\right) = A u^a + A^{\bar{a}} - A u'^a \\ &= A^{\bar{a}} + A(u^a - u'^a) = A^{\bar{a}} + A v^{\bar{a}} \end{aligned}$$

$$\begin{pmatrix} A' \\ A'^i \end{pmatrix} = \begin{pmatrix} A \\ A^i + A v^i \end{pmatrix}.$$

Example: four-velocity

$$\begin{pmatrix} 1 \\ 0'^i \end{pmatrix} = \begin{pmatrix} 1 \\ v^i \end{pmatrix}.$$

# Galilean transformation of four-covectors

$$B_a \overset{u}{\prec} (B, B_{\bar{a}}), \qquad B_a \overset{u}{\prec} (B', B'_{\bar{a}})$$

$$B' = u'^a B_a = u'^a (B\tau_a + \pi_a{}^{\bar{b}} B_{\bar{b}}) = B - v^{\bar{a}} B_{\bar{a}},$$

$$B'_{\bar{a}} = \delta_{\bar{a}}{}^b B_b = \delta_{\bar{a}}{}^b (B\tau_b + \pi_b{}^{\bar{c}} B_{\bar{c}}) = B_{\bar{a}},$$

$$(B', B'_i) = (B - v^i B_i, B^i).$$

Example: four-derivation

$$(D_{u'}, \nabla'_i) = (D_u - v^i \nabla_i, \nabla_i)$$

$$(\partial_t, \nabla_i) = (d_t - v^i \nabla_i, \nabla_i).$$

$d_t = u^a \partial_a$ comoving, substantial time derivative

# Galilean transformation of four-tensors

$$T^{ab}u^a t^b + t^{\bar{a}b} = tu^a u^b + u^a t^{\bar{b}} + t^{\bar{a}} u^b + t^{\bar{a}\bar{b}} \quad \overset{u}{\prec} \begin{pmatrix} t & t^{\bar{a}} \\ t^{\bar{b}} & t^{\bar{a}\bar{b}} \end{pmatrix}.$$

time-timelike part: $t = \tau_a \tau_b T^{ab}$
time-spacelike part: $t^{\bar{a}} = \pi^{\bar{a}}{}_c \tau_b T^{cb}$
space-timelike part: $t^{\bar{b}} = \tau_a \pi^{\bar{b}}{}_c T^{ac}$
space-spacelike part: $t^{\bar{a}\bar{b}} = \pi^{\bar{a}}{}_c \pi^{\bar{b}}{}_d T^{cd}$

$$\begin{aligned}
t' &= \tau_a \tau_b T^{ab} = t, \\
t'^{\bar{a}} &= \pi'^{\bar{a}}{}_c \tau_b T^{cb} = (\delta^a{}_c - u'^a \tau_c)(tu^c + t^{\bar{c}}) = tu^a - tu'^a + t^{\bar{a}} = t^{\bar{a}} + tv^{\bar{a}}, \\
t'^{\bar{a}\bar{b}} &= \pi'^{\bar{a}}{}_c \pi'^{\bar{b}}{}_d T^{cd} = \pi'^{\bar{a}}{}_c \pi'^{\bar{b}}{}_d (tu^c u^d + t^{\bar{c}} u^d + u^c t^{\bar{d}} + t^{\bar{c}\bar{d}}) = \\
&= tv^{\bar{a}} v^{\bar{b}} + t^{\bar{a}} v^{\bar{b}} + t^{\bar{b}} v^{\bar{a}} + t^{\bar{a}\bar{b}}.
\end{aligned}$$

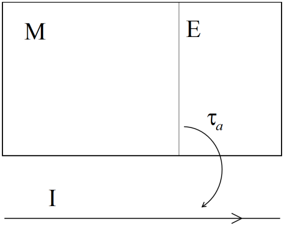
$$\begin{pmatrix} t' & t'^i \\ t'^j & t'^{ij} \end{pmatrix} = \begin{pmatrix} t & t^i + tv^i \\ t^j + tv^j & t^{ij} + t^i v^j + t^j v^i + tv^i v^j \end{pmatrix}$$

## Vectors an covectors

$$\begin{pmatrix} t' \\ x'^i \end{pmatrix} = \begin{pmatrix} t \\ x^i + v^i t \end{pmatrix}$$

Vector transformations (e.g. extensives):

$$\begin{pmatrix} A' \\ A'^i \end{pmatrix} = \begin{pmatrix} A \\ A^i + v^i A \end{pmatrix}$$




Covector transformations (e.g. derivatives):

$$\begin{pmatrix} B' & B'_i \end{pmatrix} = \begin{pmatrix} B - b_k v^k & B^i \end{pmatrix}$$

### Balances: absolute, local and substantial

$$\boxed{\partial_a A^a = 0} \qquad \longrightarrow \qquad \begin{aligned} u^a: \quad D_u A + \partial_i A^i &= d_t A + \partial_i A^i = 0, \\ u'^a: \quad D_{u'} A + \partial_i A'^i &= \partial_t A + \partial_i A'^i = 0. \end{aligned}$$

$(a,b,c \in \{0,1,2,3\})$

Transformed: $(d_t - v^i \partial_i)A + \partial_i(A^i + Av^i) = d_t A + A\partial_i v^i + \partial_i A^i = 0$

# Mass, energy and momentum

## What kind of quantity is the energy?

- Square of the relative velocity: 2nd order tensor
- Kinetic theory: trace of a contravariant second order tensor.
- Energy density and flux: additional order

## Basic field:

$$\boxed{Z^{abc} = z^{bc}u^a + z^{\bar{a}bc} : \qquad \text{mass-energy-momentum density-flux tensor}}$$

$a, b, c \in \{0,1,2,3\}, \quad \bar{a}, \bar{b}, \bar{c} \in \{1,2,3\}$

$$z^{bc} \to \begin{pmatrix} \rho & p^{\bar{b}} \\ p^{\bar{c}} & e^{\bar{b}\bar{c}} \end{pmatrix}, \qquad z^{ab} \to \begin{pmatrix} \rho & p^{\bar{a}} \\ j^{\bar{b}} & P^{\bar{a}\bar{b}} \end{pmatrix}, \qquad e = \frac{e^{\bar{b}}_{\ \bar{b}}}{2}$$

# Absolute and relative fields

$$\boxed{Z^{abc} = z^{bc}u^a + z^{\bar{a}bc} : \qquad \text{mass-energy-momentum density-flux tensor}}$$

$u$-form:

$$\begin{aligned} Z^{abc} \quad &= \quad \left(\rho u^b u^c + p^{\bar{b}} u^c + u^b p^{\bar{c}} + e^{\bar{b}\bar{c}}\right) u^a + \\ & \qquad \left(j^{\bar{a}} u^b u^c + P^{\bar{a}\bar{b}} u^c + P^{\bar{a}\bar{c}} u^b + q^{\bar{a}\bar{b}\bar{c}}\right) \end{aligned}$$

$$\begin{array}{lll} \rho & = \tau_b \tau_c z^{bc} = \tau_a \tau_b \tau_c Z^{abc}, & \text{density} \\ p^{\bar{b}} & = \pi^{\bar{b}}{}_d \tau_c z^{dc} = \tau_a \pi^{\bar{b}}{}_d \tau_c Z^{adc}, & \text{momentum density} \\ e^{\bar{b}\bar{c}} & = \pi^{\bar{b}}{}_d \pi^{\bar{c}}{}_e z^{de} = \tau_a \pi^{\bar{b}}{}_d \pi^{\bar{c}}{}_e Z^{ade}, & \text{energy density tensor} \\ j^{\bar{a}} & = \pi^{\bar{a}}{}_d \tau_b \tau_c Z^{dbc}, & \text{(self)diffusion flux} \\ P^{\bar{a}\bar{b}} & = \pi^{\bar{a}}{}_d \pi^{\bar{b}}{}_e \tau_c Z^{dec}, & \text{pressure} \\ q^{\bar{a}\bar{b}\bar{c}} & = \pi^{\bar{a}}{}_d \pi^{\bar{b}}{}_e \pi^{\bar{c}}{}_f Z^{def}. & \text{heat flux tensor} \end{array}$$

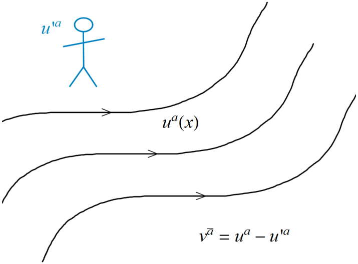
$e = \frac{1}{2} e^{\bar{a}}{}_{\bar{a}}$ energy density $\qquad$ $q^{\bar{a}} = \frac{1}{2} q^{\bar{a}\bar{b}}{}_{\bar{b}}$ heat flux

# Galilean transformation

$$\boxed{Z^{abc} = z^{bc} u^a + z^{\bar{a}bc}}$$

$$z^{bc} \overset{u}{\prec} \begin{pmatrix} \rho & p^{\bar{b}} \\ p^{\bar{c}} & e^{\bar{b}\bar{c}} \end{pmatrix}, \qquad z^{\bar{a}bc} \overset{u}{\prec} \begin{pmatrix} j^{\bar{a}} & P^{\bar{a}\bar{b}} \\ P^{\bar{a}\bar{c}} & q^{\bar{a}\bar{b}\bar{c}} \end{pmatrix}, \qquad e = \frac{e^{\bar{b}}{}_{\bar{b}}}{2}$$

Transformation rules (with relative indexes):



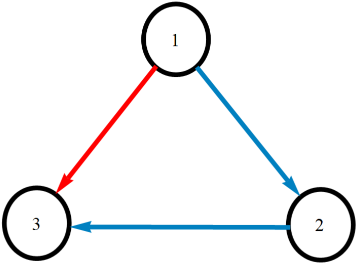

$$\begin{aligned}
\tau_a \tau_b \tau_c Z^{abc} = \hat{\rho} &= \rho, \\
\ldots = p'^i &= p^i + \rho v^i, \\
\ldots = e' &= e + p^i v_i + \rho \frac{v^2}{2}, \\
\ldots = j'^i &= j^i + \rho v^i, \\
\ldots = P'^{ij} &= P^{ij} + \rho v^i v^j + j^i v^j + p^j v^i, \\
\ldots = q'^i &= q^i + e v^i + P^{ij} v_j + p^j v_j v^i + (j^i + \rho v^i) \frac{v^2}{2}.
\end{aligned}$$

# Galiean transformation of energy

Transitivity:

$$\left.\begin{aligned} e_2 &= e_1 + p_1 v_{12} + \rho\frac{v_{12}^2}{2} \\ e_3 &= e_2 + p_2 v_{23} + \rho\frac{v_{23}^2}{2} \end{aligned}\right\} \rightarrow e_3 = e_1 + p_1 v_{13} + \rho\frac{v_{13}^2}{2}$$

$$p_2 = p_1 + \rho v_{12}, \qquad v_{13} = v_{12} + v_{23}$$



# Absolute and relative balances

## Absolute

$$\partial_a Z^{abc} = \dot{z}^{bc} + z^{bc}\partial_a u^a + \partial_a z^{\bar{a}bc} = 0$$

$$\begin{aligned}
\dot{\rho} + \rho\partial_a u^a + \partial_a j^{\bar{a}} &= 0, \\
\dot{p}^{\bar{b}} + p^{\bar{b}}\partial_a u^a + \rho\dot{u}^b + j^{\bar{a}}\partial_a u^b + \partial_a P^{\bar{a}\bar{b}} &= 0^{\bar{b}}, \\
\dot{e} + e\partial_a u^a + \partial_a q^{\bar{a}} + p^{\bar{b}}\dot{u}_b + P^{\bar{a}\bar{b}}\partial_a u_b &= 0.
\end{aligned}$$

## Relative, substantial

$$\begin{aligned}
\dot{\rho} + \rho\partial_i v^i + \partial_i j^i &= 0, \\
\dot{p}^i + p^i\partial_k v^k + \partial_k P^{ik} + \rho\dot{v}^i + j^k\partial_k v^i &= 0^i, \\
\dot{e} + e\partial_i v^i + \partial_i q^i + p^i\dot{v}_i + P^{ij}\partial_i v_j &= 0.
\end{aligned}$$

# Thermodynamics. Gibbs relation I.

$\boxed{ds = Y_{bc} dz^{bc}}$ $Y_{bc}$ chemical potential-thermovelocity-temperature cotensor

## Physical definitions

$$Y_{bc} \overset{u}{\prec} \begin{pmatrix} y & y_{\overline{b}} \\ y_{\overline{c}} & y_{\overline{c}\overline{b}} \end{pmatrix} = \frac{\beta}{2} \begin{pmatrix} -2\mu & -w_{\overline{b}} \\ -w_{\overline{c}} & \delta_{\overline{b}\overline{c}} \end{pmatrix},$$

Transformation rules

$$\begin{aligned} \beta' &= \beta, \\ w_i' &= w_i + v_i, \qquad \text{like a vector!} \\ \mu' &= \mu - w_i v^i - \frac{v^2}{2}. \end{aligned}$$

Calculation with classical transformation matrix.

# Thermodynamics. Gibbs relation II.

Absolute Gibbs relation: $\boxed{ds = Y_{bc} dz^{bc}}$

Absolute extensivity condition: $\boxed{S^a = Y_{bc} Z^{abc} + p^a}$

## Absolute and relative

Pressure decomposition: $p^a = \beta p(u^a + w^{\bar{a}})$

$$\begin{aligned} S^a = Y_{bc} Z^{abc} + p^a \quad &\rightarrow \quad Ts = e + p - \mu\rho - w_i p^i, \\ &\rightarrow \quad Ts^i = q^i - \mu j^i - P^{ij} w_j + p w^i, \\ ds = Y_{bc} dz^{bc} \quad &\rightarrow \quad de = Tds + \mu d\rho + w_i dp^i + (\rho w_i - p_i) dv^i. \end{aligned}$$

Relative Gibbs relation is Galilean invariant if the inertial reference frame changes.

# Thermostat(odynam)ics.

Gibbs relation: $de = Tds + \mu d\rho + w_i dp^i + (\rho w_i - p_i)dv^i$

## Maxwell relations

$s(e, \rho, p^i, v^i)$

$$\frac{\partial s}{\partial p^i} = \frac{w_i}{T}, \qquad \frac{\partial s}{\partial v^i} = \frac{\rho w_i - p_i}{T}$$

$$\frac{\partial^2 s}{\partial v^i p^j} = \frac{\partial^2 s}{\partial p^i v^j} = \boxed{\frac{\partial w_i}{\partial v^j} = \delta_{ij} - \rho \frac{\partial w_i}{\partial p^j}}$$

Solution:

$$w_i = \frac{p_i}{\rho} + A_{ij}\left(v^j + \frac{p^j}{\rho}\right) + \overline{w}_i$$

Galilean invariant(!) part:

$$\boxed{p_i = \rho w_i}$$

# Termodynamics III. Entropy balance.

$$\partial_a S^a = \partial_a(su^a + s^{\bar{a}}) = \sigma \geq 0, \text{ condition}: \partial_a Z^{abc} = 0$$

## Entropy production

$$\begin{aligned}
\partial_a S^a &= \dot{s} + s\partial_a u^a + \partial_a s^{\bar{a}} \\
&= \dots \\
&= -(j^{\bar{a}} - \rho w^{\bar{a}})\partial_a \left(\beta\mu + \beta\frac{w^2}{2}\right) + \\
&\quad \left(q^{\bar{a}} - w^{\bar{a}}(e - p^{\bar{b}} w_{\bar{b}}) + (j^{\bar{a}} - \rho w^{\bar{a}})\frac{w^2}{2} - P^{\bar{a}\bar{b}} w_{\bar{b}}\right)\partial_a\beta - \\
&\quad \beta\left(P^{\bar{a}}_{\bar{b}} + w^{\bar{a}}(\rho w_{\bar{b}} - p_{\bar{b}}) - j^{\bar{a}} w_{\bar{b}} - p\delta^{\bar{a}}_{\bar{b}}\right)\partial_a(u^b + w^{\bar{b}}) \geq 0
\end{aligned}$$

# Entropy production II.

$$
\begin{aligned}
\dot\rho + \rho\partial_i v^i + \partial_i j^i &= 0,\\
\dot p^i + p^i\partial_k v^k + \partial_k P^{ik} + \rho\dot v^i + j^k\partial_k v^i &= 0^i,\\
\dot e + e\partial_i v^i + \partial_i q^i + p^i\dot v_i + P^{ij}\partial_i v_j &= 0.
\end{aligned}
$$

$$
\begin{aligned}
\Sigma &= -(j^i - \rho w^i)\partial_i\left(\beta\mu + \beta\frac{w^2}{2}\right) + \\
&\left(q^i - w^i(e - p^j w_j) + (j^i - \rho w^i)\frac{w^2}{2} - P^{ij}w_j\right)\partial_i\beta - \\
&\beta\left(P^i_j + w^i(\rho w_j - p_j) - j^i w_j - p\delta^i_j\right)\partial_i(v^j + w^j) \geq 0
\end{aligned}
$$

Variables: $\rho, p^i, e$

Constitutive functions: $j^i, P^{ij}, q^i$, $\boxed{v^i?}$ flow-frame

Equation of state: $\mu, T, w^i$

# Classical theory

Eos: $w^i = \frac{p^i}{\rho}$

Flow frame: $A^{\bar{a}} = 0$ if $u^a = \frac{A^a}{\tau_a A^a}$

**Thermo-frame**

$$w^{\bar{a}} = 0 \quad \rightarrow \quad p^{\bar{a}} = 0$$

$$\begin{aligned}
\dot{\rho} + \rho\partial_i v^i + \partial_i j^i &= 0, \\
\rho\dot{v}^i + \partial_k P^{ik} + j^k\partial_k v^i &= 0^i, \\
\dot{e} + e\partial_i v^i + \partial_i q^i + P^{ij}\partial_i v_j &= 0.
\end{aligned}$$

$$-j^i\partial_i\frac{\mu}{T} + q^i\partial_i\frac{1}{T} - \frac{1}{T}(P^{ij} - p\delta^{ij})\partial_i v_j \geq 0$$

Flow-frame: hidden Galilean invariance

# Constitutive theory

$$-j^i\partial_i\frac{\mu}{T} + q^i\partial_i\frac{1}{T} - \frac{1}{T}(P^{ij} - p\delta^{ij})\partial_i v_j \geq 0$$

| | Diffusion | Thermal | Mechanical |
|---|---|---|---|
| Force | $-\partial_i\frac{\mu}{T}$ | $\partial_i\frac{1}{T}$ | $\partial_i v_j$ |
| Flux | $j^i$ | $q^i$ | $-\frac{1}{T}\left(P^{ij} - p\delta^{ij}\right)$ |

$$\begin{aligned}
\dot{\rho} + \rho\partial_i v^i + \partial_i j^i &= 0, \\
\rho\dot{v}^i + \partial_k P^{ik} + j^k\partial_k v^i &= 0^i, \\
\dot{e} + e\partial_i v^i + \partial_i q^i + P^{ij}\partial_i v_j &= 0.
\end{aligned}$$

(Self)-diffusion : not Brenner like

# Conclusions

- Fluid mechanics, thermodynamics and entropy production are absolute: independent of reference and flow-frames. Thermodynamic fluxes and forces are Galilean invariant.
- Transformation rules can be calculated.
- Best if thermodynamics defines the flow. Thermovelocity is given by an equation of state ($p^i = \rho w^i$). (Self)diffusion cannot be eliminated.
- Linear asymptotic stability of homogeneous equilibrium.
- Kinetic theory? (Ruggeri-Sugiyama)
- Special relativistic hydro? Fluxes and forces?
- Hyperbolicity, symmetry? (Godunov)
- Material frame indifference

# Thank you for the attention !

More details are in :

http://arxiv.org/pdf/1508.00121v2