Objectivity and constitutive modelling

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– Objectivity – contra Noll
– Kinematics – rate definition
– Frame independent Liu procedure
– Second and third grade solids (waves)
Problems with the traditional formulation

Rigid rotating frames:

\[ \hat{t} = t, \quad \hat{x} = h(t) + Q(t)x \quad \text{Noll (1958)} \]

Four-transformations, four-Jacobian:

\[ C^a = J^a_b c^b, \quad \text{where} \quad J^a_b = \frac{\partial \hat{x}^a}{\partial x^b} = \begin{pmatrix} 1 & 0 \\ \dot{h} + \dot{Q}x & Q \end{pmatrix} \]

\[
\begin{pmatrix} \hat{c}^0 \\ \hat{c} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dot{h} + \dot{Q}x & Q \end{pmatrix} \begin{pmatrix} c^0 \\ c \end{pmatrix} = \begin{pmatrix} c^0 \\ (\dot{h} + \dot{Q}x)c^0 + Qc \end{pmatrix}
\]
Consequences:

\[ \Rightarrow \] four-velocity is an objective vector,
\[ \Rightarrow \] time cannot be avoided,
\[ \Rightarrow \] transformation rules are not convenient,
\[ \Rightarrow \] why rigid rotation?

\[ \Rightarrow \] arbitrary reference frames
\[ \Rightarrow \] reference frame INDEPENDENT formulation!
What is non-relativistic space-time?

\[ M = \mathbb{R}^3 \times \mathbb{R} \]

\[ M = \mathbb{E} \times \mathbb{I} \]
The aspect of spacetime:

**Special relativity**: no absolute time, no absolute space, 
exists absolute spacetime; three-vectors → four-vectors, 
\(x(t) \rightarrow \text{world line} \) (curve in the 4-dimensional spacetime).

**The nonrelativistic case** (Galileo, Weyl): exists absolute time, 
no absolute space, spacetime needed \((t' = t, x' = x - vt)\)

4-vectors: \(
\begin{pmatrix}
t \\
x
\end{pmatrix}
\), world lines: \(x(t) \rightarrow \begin{pmatrix} t \\ x(t) \end{pmatrix}\)

(4-coordinates w.r.t. an inertial coordinate system K)

The spacelike 4-vectors \(
\begin{pmatrix} 0 \\ x \end{pmatrix}
\) are absolute: form a 3-dimensional Euclidean vector space.
World lines of a body – material manifold?

\[
\ddot{\mathbf{Y}}^a_b = \partial_b \ddot{\mathbf{x}}^a = \begin{pmatrix} t \\ \dddot{x}^i(t, X_i) \end{pmatrix} (\partial_t, \partial_j) = \begin{pmatrix} 1 & 0_j \\ \partial_t \dddot{x}^i & \partial_j \dddot{x}^i \end{pmatrix} = \begin{pmatrix} 1 & 0_j \\ \vec{v}^i & \vec{F}^i_j \end{pmatrix}
\]

\( a, b \in \{0, 1, 2, 3\}, \quad i, j \in \{1, 2, 3\} \)

deformation gradient
Objective derivative of a vector

\[ \tilde{\partial}_a \hat{c}^b \sim \left( \frac{c^0}{(F^{-1})^j_k \tilde{c}^k} \right) (\tilde{\partial}_i, \tilde{\partial}_i) = \begin{pmatrix} \frac{d}{dt} \hat{c}^0 \\ \frac{d}{dt} \left( (F^{-1})^j_k \tilde{c}^k \right) \\ \tilde{\partial}_i \left( (F^{-1})^j_k \tilde{c}^k \right) \end{pmatrix} = \begin{pmatrix} \hat{\nabla}^0 \\ \nabla \hat{c}^0 \\ F^{-1}(\hat{c} - \nabla v \cdot \tilde{c}) \\ F^{-1}(\tilde{\nabla} \hat{c} - \nabla F \cdot \tilde{c}) \end{pmatrix} \]

Upper convected

Tensorial property (order, co-) determines the form of the derivative.
Kinematics

Spacetime-allowed quantities:
• the velocity field \( v \) [the 4-velocity field \( \left( \begin{array}{c} 1 \\ v \end{array} \right) \)],
• its integral curves (world lines of material points),
\[ \partial_i v^j, \frac{1}{2}(\partial_i v^j + \partial_j v^i), \frac{1}{2}(\partial_i v^j - \partial_j v^i) \]

The problem of reference time: is the continuum completely relaxed, undisturbed, totally free at \( t_0 \), at every material point?

In general, not \( \rightarrow \) incorrect for reference purpose.

The physical role of a strain tensor: to quantitatively characterize not a change but a state:
not to measure a change since a \( t_0 \) but to express the difference of the current local geometric status from the status in a fully relaxed, ideally undisturbed state.
What one can have in general for a strain tensor: a rate equation + an initial condition (which is the typical scheme for field quantities, actually)

Spacetime requirements → the change rate of strain is to be determined by \( \partial_i v^j \)

Example: instead of \( E^{Cauchy} = (\nabla_R u)^S \)
from now on:

\[
\dot{E}^{Cauchy} = (\nabla_R v)^S + \text{initial condition}
\]

(e.g. knowing the initial stress and a linear elastic const. rel.)

Remark: the natural strain measure rate equation is

\[
\dot{A} = (\nabla_R v)A + A(\nabla_R v)^T \quad \text{generalized left Cauchy-Green}
\]
Frame independent constitutive theory:

⇒ variables:
   spacetime quantities (v!)
   body related rate definition of the deformation

⇒ balances:
   frame independent spacetime relations

⇒ pull back to the body (reference time!)

Entropy flux is constitutive!

Question: fluxes?
Constitutive theory of a second grade elastic solid

Balances:

\[
\lambda: \quad \rho_0 \dot{e} + \tilde{\partial}_i w^i = 0, \\
\lambda_i: \quad \rho_0 \dot{\nu}^i - \tilde{\partial}_j T_{ij} = 0^i, \\
\Lambda^j_i: \quad \dot{F}^i_{\ j} - \tilde{\partial}_j \nu^i = 0^i_j.
\]

Remark:

\[
\tilde{Y}^a_b = \tilde{\partial}_b \tilde{x}^a = \begin{pmatrix} 1 & 0_j \\ \tilde{\nu}^i & \tilde{F}^{i}_{\ j} \end{pmatrix}, \quad \tilde{\partial}_k \tilde{Y}^a_b = \tilde{\partial}_k \tilde{x}^a = \begin{pmatrix} 0_k & 0_{kj} \\ \tilde{\partial}_k \tilde{\nu}^i & \tilde{\partial}_k \tilde{F}^{i}_{\ j} \end{pmatrix}
\]

Constitutive space: \( e, \tilde{\partial}_i e, \nu^i, \tilde{\partial}_j \nu^i, F^i_{\ j}, \tilde{\partial}_k F^i_{\ j} \)

Constitutive functions: \( w^i, T^{ij}, s, J^i \)
Second Law:

\[ \rho_0 \dot{s} + \tilde{\partial}_i J^i - \lambda (\rho_0 \dot{e} + \tilde{\partial}_i w^i) - \lambda_i (\rho_0 \dot{v}^i - \tilde{\partial}_j T^{ij}) - \Lambda^j_i (\dot{F}^i_j - \tilde{\partial}_i v^j) \geq 0, \]

Consequences:

Function restrictions:

\[ w^i (e, v^i, F^i_j), T^{ij} (e, v^i, F^i_j), \quad s(e, v^i, F^i_j), \]

\[ J^i = \partial_e s w^i - \partial_{v^i} s T^{ji} + K(e, v^i, F^i_j) \]

- Internal energy \( s(e - \frac{v^2}{2}, F^i_j) \)

Dissipation inequality:

\[ \mathbf{q} \cdot \nabla_R \frac{1}{T} + \frac{1}{T} \left( T - \rho_0 \tilde{\partial}_{F^i_j} f \right) \hat{\mathbf{F}} \geq 0 \]
Constitutive theory of a third grade elastic solid

Balances:

\[ \lambda : \quad \rho_0 \dot{e} + \tilde{\partial}_i w^i = 0, \]
\[ \lambda_i : \quad \rho_0 \dot{v}^i - \tilde{\partial}_j T^{ij} = 0^i, \quad \text{Constitutive functions:} \]
\[ \hat{\lambda}_i^j : \quad \rho_0 \tilde{\partial}_j \dot{v}^i - \tilde{\partial}_{jk} T^{ik} = 0^i_j, \]
\[ \Lambda^j_i : \quad \dot{F}^i_j - \tilde{\partial}_j v^i = 0^i_j, \]
\[ \hat{\Lambda}^{jk}_i : \quad \tilde{\partial}_k \dot{F}^i_j - \tilde{\partial}_{kj} v^i = 0^i_{jk} \]

Constitutive space: \( e, \tilde{\partial}_i e, v^i, \tilde{\partial}_j v^i, \tilde{\partial}_{jk} v^i, F^i_j, \tilde{\partial}_k F^i_j, \tilde{\partial}_{kl} F^i_j \)

Second Law:

\[ \rho_0 \dot{s} + \tilde{\partial}_i J^i - \lambda (\rho_0 \dot{e} + \tilde{\partial}_i w^i) - \lambda_i (\rho_0 \dot{v}^i - \tilde{\partial}_j T^{ij}) - \Lambda^j_i (\dot{F}^i_j - \tilde{\partial}_i v^i) - \hat{\lambda}_i^j (\rho_0 \tilde{\partial}_j \dot{v}^i - \tilde{\partial}_{jk} T^{ik}) - \hat{\Lambda}^{jk}_i (\tilde{\partial}_k \dot{F}^i_j - \tilde{\partial}_{kj} v^i) \geq 0, \]
Consequences:

Function restrictions:

$$T^{ij}(e, v^i, \tilde{\partial}_j v^i, F^{i_j}, \tilde{\partial}_k F^{i_j}), \quad s(e, v^i, \tilde{\partial}_j v^i, F^{i_j}, \tilde{\partial}_k F^{i_j}),$$

$$J^i = \partial_e s w^j - \partial_{v^j} s T^{ji} + \ldots + K(e, v^i, \tilde{\partial}_j v^i, F^{i_j}, \tilde{\partial}_k F^{i_j})$$

- Internal energy

$$s(e - \frac{v^2}{2} - \frac{\alpha}{2} \text{Tr}(\dot{\mathbf{F}} \cdot \dot{\mathbf{F}}) - \ldots, F^{i_j})$$

Dissipation inequality:

$$\mathbf{q} \cdot \nabla \frac{1}{T} + \frac{1}{T} \left( \mathbf{T} - \alpha \nabla_R \nabla_R \cdot \mathbf{T} - \rho_0 \partial_{F^{i_j}} f + \rho_0 \nabla_R \cdot \left( \partial_{\tilde{\partial}_k F^{i_j}} f \right) \right); \quad \dot{\mathbf{F}} \geq 0$$

Double stress
Waves- 1d

Quadratic energy – without isotropy

\[ f(F^i_j, \tilde{\partial}_k F^i_j) = \frac{\mu}{2} F^i_j F^j_i + \frac{\hat{\mu}}{2} \tilde{\partial}_k F^i_j \tilde{\partial}^k F^j_i + ... \]

\[ (T - \alpha \nabla \cdot T - \rho_0 \partial F^i_j f + \rho_0 \tilde{\partial}_k (\tilde{\partial}_k F^i_j f)) : \dot{F} \geq 0 \]

\[ T - \mu F - \alpha \tilde{\partial}_{xx} T + \hat{\mu} \tilde{\partial}_{xx} F - \eta \tilde{\partial}_i F = 0, \]
\[ \rho_0 \tilde{\partial}_{tt} F - \tilde{\partial}_{xx} T = 0. \]

- Stable
- Dispersion relation

\[ c^2 = \left( \frac{\omega}{k} \right)^2 = \frac{k^2 \mu + \hat{\mu} k^2}{\rho_0 (1 + \alpha k^2)} \]
Summary

- Noll is wrong.

- Aspects of non-relativistic spacetime cannot be avoided.

- Deformation and strain – wordline based rate definition.

- Gradient elasticity theories are valid.
Thank you for your attention!
Thermodynamics - Mechanics

**Real world** – Second Law is valid

**Ideal world** – there is no dissipation

**Impossible world** – Second Law is violated