Can material time derivative be objective?

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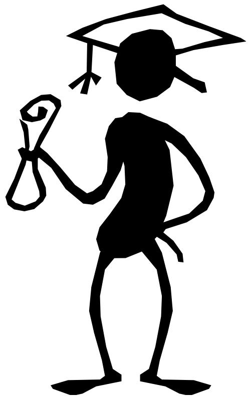
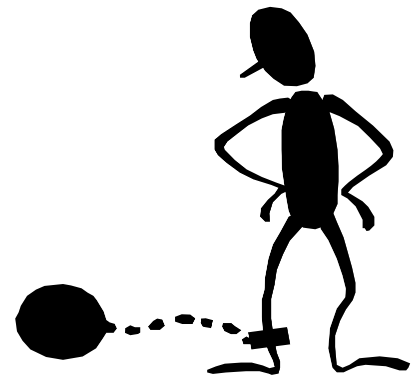


Classical
irreversible
thermodynamics



Local equilibrium (~ there is no microstructure)

Beyond local equilibrium:

Nonlocality in space
(structures)

Nonlocality in time
(memory and inertia)

constitutive space

Basic state space: $\mathbf{a} = (\ldots\ldots)$

Nonlocality in space
(structures)

Nonlocality in time
(memory and inertia)

constitutive space
(weakly nonlocal)

???

$$\nabla \mathbf{a}$$

$$(\nabla^2 \mathbf{a}, \ldots, \nabla^n \mathbf{a})$$

$$\dot{\mathbf{a}}$$

$$(\dot{\mathbf{a}}, \ldots, \mathbf{a}^{(n\cdot)})$$

Nonlocality in spacetime

Rheology

Jaumann (1911)
Oldroyd (1949, …)
…

Thermodynamic theory:

Kluitenberg (1962, …),
Kluitenberg, Ciancio and Restuccia (1978, …)

Verhás (1977, …, 1998)

Thermodynamic theory with co-rotational time derivatives.

Experimental proof and prediction:

- viscometric functions of shear hysteresis
- instability of the flow !!

Material frame indifference

Noll (1958), Truesdell and Noll (1965)
Müller (1972, …) (kinetic theory)
Edelen and McLennan (1973)
Bampi and Morro (1980)
Ryskin (1985, …)
Lebon and Boukary (1988)
Massoudi (2002) (multiphase flow)
Speziale (1981, …, 1998), (turbulence)
Murdoch (1983, …, 2005) and Liu (2005)
Muschik (1977, …, 1998), Muschik and Restuccia (2002)
……..

Objectivity

About objectivity:

$$\hat{t} = t, \qquad \hat{\mathbf{x}} = \mathbf{h}(t) + \mathbf{Q}(t)\mathbf{x}$$ Noll (1958)

$$\hat{x}^0 = x^0, \quad \hat{x}^\alpha = h^\alpha + Q^\alpha{}_\beta x^\beta, \quad \alpha, \beta = 1,2,3$$

$$\hat{x}^i = \hat{x}^i(x), \qquad i = 0,1,2,3$$

$c^i = (c^0, \mathbf{c})$ is a four dimensional objective vector, if

$$\hat{c}^i = \hat{J}^i{}_j c^j, \qquad \text{where} \qquad \hat{J}^i{}_j = \frac{\partial \hat{x}^i}{\partial x^j}$$

$$\hat{J} = \begin{pmatrix} 1 & 0 \\ \dot{\mathbf{h}} + \dot{\mathbf{Q}}\mathbf{x} & \mathbf{Q} \end{pmatrix}$$

$$\begin{pmatrix} \hat{c}^0 \\ \hat{\mathbf{c}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dot{\mathbf{h}} + \dot{\mathbf{Q}}\mathbf{x} & \mathbf{Q} \end{pmatrix} \begin{pmatrix} c^0 \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} c^0 \\ (\dot{\mathbf{h}} + \dot{\mathbf{Q}}\mathbf{x})c^0 + \mathbf{Q}\mathbf{c} \end{pmatrix}$$

Spec. 1: $c^0 = 0 \quad \Rightarrow \quad \hat{\mathbf{c}} = \mathbf{Q}\mathbf{c}$

Spec. 2: motion $\mathbf{r}(t) \quad \Rightarrow \quad \dot{\mathbf{r}} = \mathbf{v}$

$$\hat{\mathbf{r}} = \mathbf{h} + \mathbf{Q}\mathbf{r} \quad \Rightarrow \quad \dot{\hat{\mathbf{r}}} = \hat{\mathbf{v}} = \dot{\mathbf{h}} + \dot{\mathbf{Q}}\mathbf{r} + \mathbf{Q}\dot{\mathbf{r}}$$

$(t, \mathbf{r}(t)) \quad \Rightarrow \quad (1, \mathbf{v}(t))$ is an objective four vector

$$\begin{pmatrix} 1 \\ \hat{\mathbf{v}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dot{\mathbf{h}} + \dot{\mathbf{Q}}\mathbf{x} & \mathbf{Q} \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{v} \end{pmatrix}$$

Covariant derivatives:

as the spacetime is flat there is a distinguished one.

$$Da \sim D_i a = \partial_i a$$ covector field

$$Dc \sim D_j c^i = \partial_j c^i + \Gamma^i_{jk} c^k$$ mixed tensor field

The coordinates of the covariant derivative of a vector field do not equal the partial derivatives of the vector field if the coordinatization is not linear.

If $\hat{x}$ are inertial coordinates, the Christoffel symbol with respect to the coordinates x has the form:

$$\Gamma^i_{jk} = \frac{\partial^2 \hat{x}^m}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial \hat{x}^m} = -\frac{\partial^2 x^i}{\partial \hat{x}^m \partial \hat{x}^l} \frac{\partial \hat{x}^m}{\partial x^j} \frac{\partial \hat{x}^l}{\partial x^k}$$

$$\hat{t} = t, \qquad \hat{\mathbf{x}} = \mathbf{h}(t) + \mathbf{Q}(t)\mathbf{x}$$

$$\hat{x}^0 = x^0, \quad \hat{x}^\alpha = h^\alpha + Q^\alpha{}_\beta x^\beta, \quad \alpha, \beta = 1,2,3$$

$$\Downarrow$$

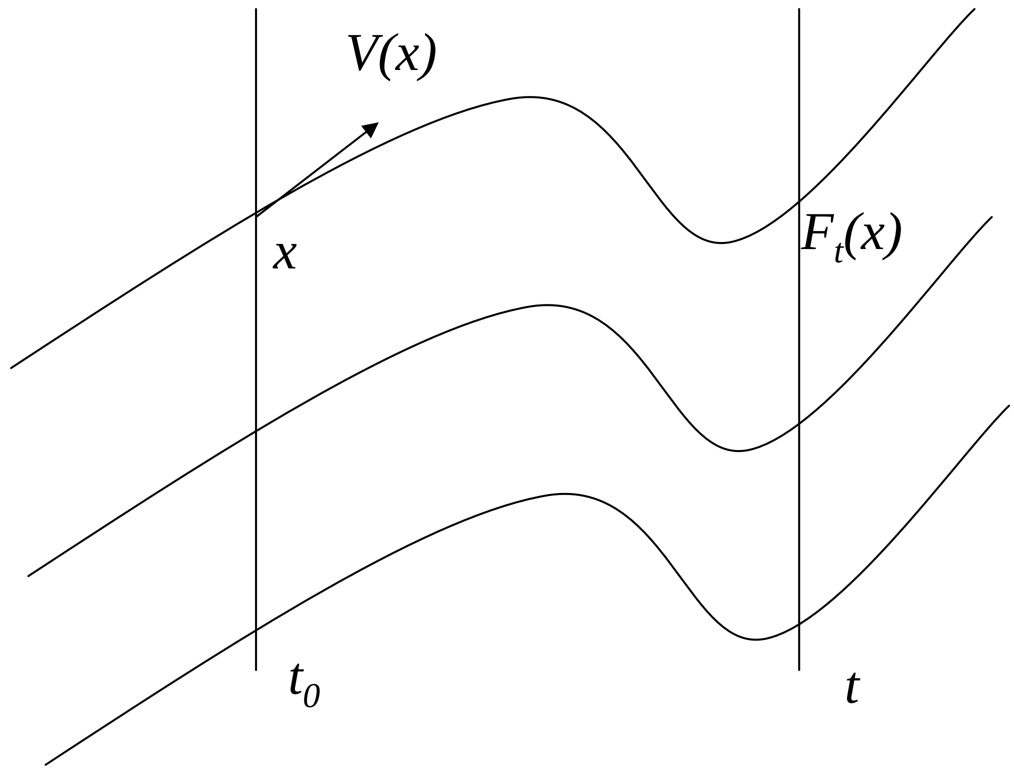
$$\Gamma^0_{jk} = 0, \quad \Gamma^\alpha_{0\beta} = \Gamma^\alpha_{\beta 0} = \Omega^\alpha{}_\beta, \quad \Gamma^\alpha_{\beta\gamma} = 0,$$

where $\mathbf{\Omega} = \mathbf{Q}^{-1}\dot{\mathbf{Q}}$ is the angular velocity of the observer

$$\Gamma^\alpha_{00} = (\mathbf{Q}^{-1}(\ddot{\mathbf{h}} + \ddot{\mathbf{Q}}\mathbf{x}))^\alpha = (\mathbf{Q}^{-1}\ddot{\mathbf{h}} + (\dot{\mathbf{\Omega}} + \mathbf{\Omega}\mathbf{\Omega})\mathbf{x}))^\alpha$$

Material time derivative:

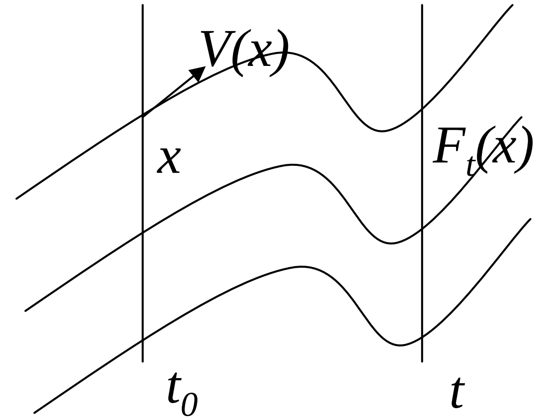
Flow generated by
a vector field V.



$F_t(x)$ is the point at time t of the integral curve V passing through x.

$\Phi(F_t(x))$ is the change of Φ along the integral curve.

$$\frac{d}{dt}\Phi(F_t(x)) = D_V \Phi(F_t(x))$$



$D_V \Phi = V \cdot D\Phi$ is the covariant derivative of Φ according to *V*.

$$V = (1, \mathbf{v})\,!$$

Spec. 1: Φ is a scalar

$$D_V a = V^j D_j a = V^j \partial_j a$$

$$D_V a = (\partial_0 + \mathbf{v} \cdot \nabla) a = \dot{a}$$ substantial time derivative

Spec. 2: $\Phi = c = (0, \mathbf{c})$ is a spacelike vector field

$$(D_V c)^i = V^j D_j c^i = V^j (\partial_j c^i + \Gamma^i_{jk} c^k)$$

$$\Downarrow$$

$$D_V \mathbf{c} = (\partial_0 + \mathbf{v} \cdot \nabla + \mathbf{\Omega})\mathbf{c}$$

The material time derivative of a vector – even if it is spacelike – is not given by the substantial time derivative.

Jaumann, upper convected, etc… derivatives:

In our formalism: ad-hoc rules to eliminate the Christoffel symbols.
For example:

$$V^j D_j c^i = V^j(\partial_j c^i + \Gamma^i_{jk} c^k)$$

$$c^j D_j V^i = c^j(\partial_j V^i + \Gamma^i_{jk} V^k)$$

$$\Big\Downarrow \ \Gamma^i_{jk} = \Gamma^i_{kj}$$

$$V^j D_j c^i - c^j D_j V^i = V^j \partial_j c^i - c^j \partial_j V^i$$

for a spacelike vector $(0, \mathbf{c})$

$$V^j \partial_j c^i - c^j \partial_j V^i = (\partial_0 + \mathbf{v}\cdot\nabla)\mathbf{c} - \mathbf{c}\cdot\nabla\mathbf{v}$$

upper convected (contravariant) time derivative

One can get similarly Jaumann, lower convected, etc…

Conclusions:

– Objectivity has to be extended to a four dimensional setting.

– Four dimensional covariant differentiation is fundamental in non-relativistic spacetime. The essential part of the Christoffel symbol is the angular velocity of the observer.

– Partial derivatives are not objective. A number of problems arise from this fact.

– Material time derivative can be defined uniquely. Its expression is different for fields of different tensorial order.

$$\text{space} + \text{time} \neq \text{spacetime}$$


Thank you for your attention.