# Weakly nonlocal fluid mechanics

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- One component fluid mechanics - quantum (?) fluids
- Quantum potential
  - Why Fisher information?
- Two component fluid  mechanics – sand (?)
- Conclusions

# *Why nonequilibrium thermodynamics?*

Thermodynamics $\neq$ science of temperature

Thermodynamics $\neq$ science of macroscopic energy changes

Thermodynamics (?) $=$ general framework of any macroscopic (?) continuum (?) theories

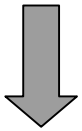
General framework:

- fundamental balances
- objectivity - frame indifference
- Second Law $\Rightarrow$ reversibility – special limit

*Phenomenology* – minimal or no microscopic information

universality

*Second Law* – "super-principle"

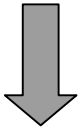


– valid for all kind of dynamics

– like symmetries

*Beyond local equilibrium* – memory and inertia

*Beyond local state* – nonlocality



<u>weak</u>

– short range - not gravity

– higher order gradients

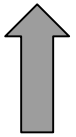
# Non-equilibrium thermodynamics

$$\dot{\mathbf{a}} + \nabla \cdot \mathbf{j}_a = \sigma_a \qquad \text{basic balances} \qquad \mathbf{a} = (\rho, \rho\mathbf{v}, \ldots)$$

– basic state: $\mathbf{a}$

– constitutive state: $C = (\mathbf{a}, \dot{\mathbf{a}}, \nabla\mathbf{a}, \ldots)$

– constitutive functions: $\mathbf{j}_a(C)$

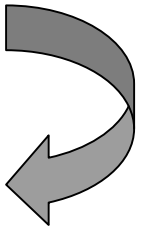


weakly nonlocal

<u>Second law:</u>

$$\dot{s}(C) + \nabla \cdot \mathbf{j}_s(C) = \sigma_s \geq 0$$

Constitutive theory

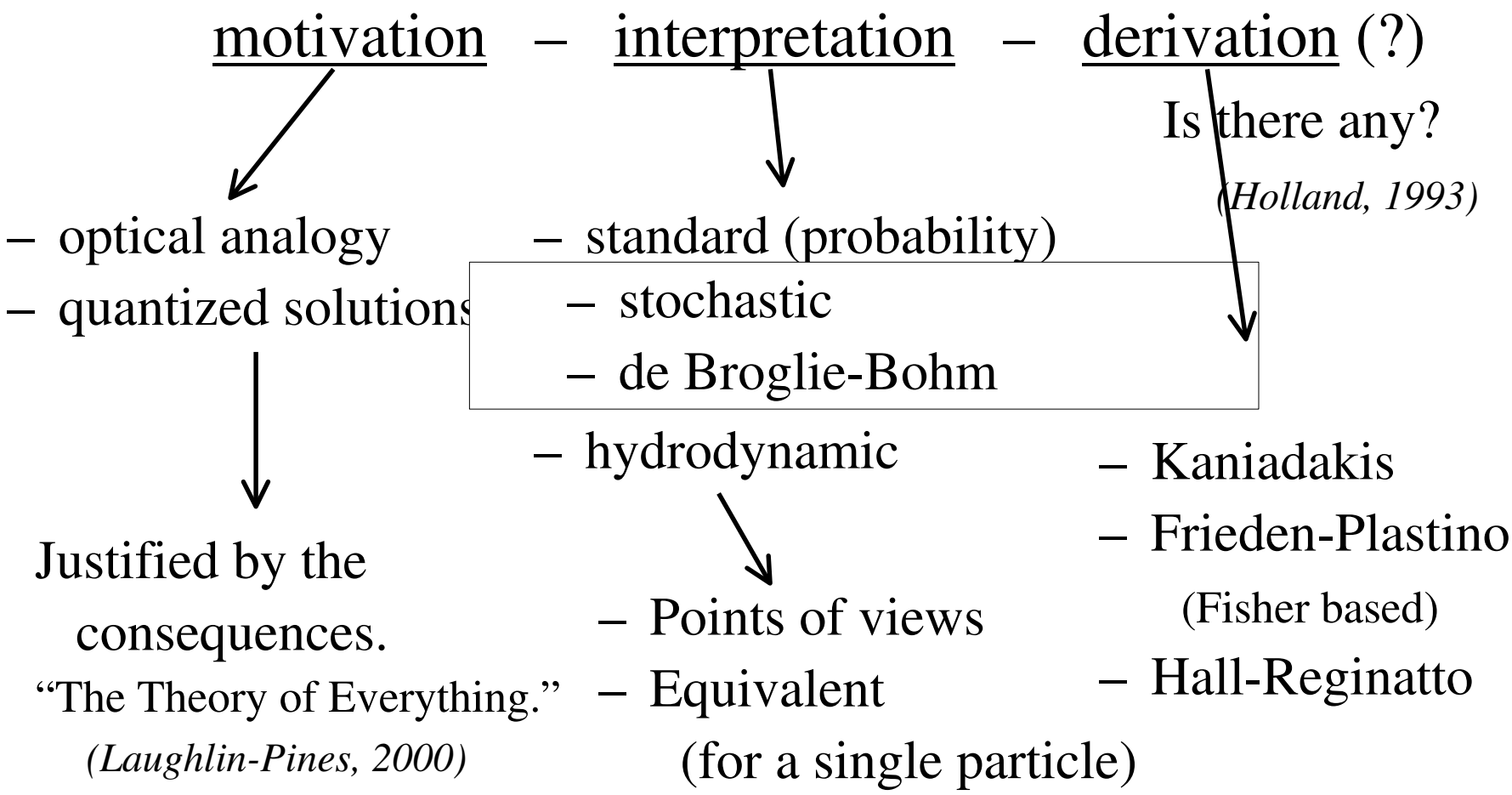


(universality)

Method: Liu procedure, Lagrange-Farkas multipliers

Special: irreversible thermodynamics

***Origin of quantum mechanics:***



Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\Delta\psi+V(x)\psi$$

Madelung transformation: $\psi = R\,e^{iS}$

$$\dot{\rho}+\rho\nabla\cdot\mathbf{v}=0$$

$$\rho := R^2 \qquad \mathbf{v} := \frac{\hbar}{m}\nabla S$$

de Broglie-Bohm form:

$$\dot{\mathbf{v}}=-\nabla(U_{QM}+V)$$

$$U_{QM}=-\frac{\hbar^2}{2m^2}\frac{\Delta R}{R}$$

Hydrodynamic form:

$$\rho\dot{\mathbf{v}}+\nabla\cdot\mathbf{P}_{QM}=-\rho\nabla V$$

Fundamental questions in quantum mechanics:

– Why we need variational principles?
(What is the physics behind?)

– Why we need a wave function?
(What is the physics behind?)

– Where is frame invariance (objectivity)?

# One component weakly nonlocal fluid

$$\dot{\rho} + \rho \nabla \cdot \mathbf{v} = 0$$
$$\rho \dot{\mathbf{v}} + \nabla \cdot \mathbf{P}(C) = \mathbf{0}$$
$$\rho \dot{s}(C) + \nabla \cdot \mathbf{j}_s(C) \geq 0$$

$(\rho, \mathbf{v})$ basic state

$C_{wnl} = (\rho, \nabla\rho, \nabla\nabla\rho, \mathbf{v}, \nabla\mathbf{v})$

constitutive state

$s(C), \mathbf{j}_s(C), \mathbf{P}(C)$

⇧

constitutive functions

Liu procedure (Farkas's lemma):

$$s(\rho, \mathbf{v}, \nabla\rho) = s_e(\rho, \nabla\rho) - \frac{\mathbf{v}^2}{2} \qquad \mathbf{j}_s = -\mathbf{v} \cdot \mathbf{P} + \ldots$$

$\mathbf{P}^r$ reversible pressure

$$\sigma_s = -\left( \mathbf{P} - \rho^2 \left[ \quad - \partial_\rho s \right] \mathbf{I} \quad \right) : \nabla \mathbf{v} \geq 0$$

$$\mathbf{P}^v = \mathbf{P} - \mathbf{P}^r$$

➡ Potential form: $\nabla \cdot \mathbf{P}^r = \rho \nabla U_Q$

➡ $U_Q = -\partial_\rho (\rho s_e) + \nabla \cdot (\rho \partial_{\nabla \rho} s_e)$

Euler-Lagrange form

⬇

Variational origin

# Schrödinger-Madelung fluid

(Fisher entropy)

$$s_{SchM}(\rho, \mathbf{v}, \nabla\rho) = -\frac{\nu_{SchM}}{2}\left(\frac{\nabla\rho}{2\rho}\right)^2 - \frac{\mathbf{v}^2}{2}$$

$$\mathbf{P}^{\mathrm{r}}_{\mathrm{SchM}} = -\frac{1}{8}\left(\Delta\rho\mathbf{I} + \nabla^2\rho - \frac{2\nabla\rho\otimes\nabla\rho}{\rho}\right)$$

$\Psi = \sqrt{\rho}e^{-i\nabla\mathbf{v}}$

⇒ Bernoulli equation ⇒ Schrödinger equation

# Landau fluid

$$s_{Lan}(\rho, \nabla\rho) = -\nu_{Lan}\frac{(\nabla\rho)^2}{2}$$

$$\mathbf{P}^r_{Lan} = -\nu_{Lan}\frac{\rho^2}{2}\left(\Delta\rho\mathbf{I} + \nabla^2\rho\right)$$

$$U_{Lan} = -\nu_{Lan}\frac{1}{2}\left(\rho\Delta\rho + \Delta\frac{\rho^2}{2}\right)$$

Alternate fluid

$$s_{Alt}(\rho, \nabla\rho) = -\nu_{Alt}\frac{(\nabla\rho)^2}{\rho}$$

$$\mathbf{P}^r_{Alt} = -\frac{\nu_{Alt}}{4}\left(\rho(\Delta\rho\mathbf{I} + \nabla^2\rho) - \nabla\rho\circ\nabla\rho\right)$$

$$U_{Alt} = -\frac{\nu_{Alt}}{2}\Delta\rho$$

Korteweg fluids:

$$\mathbf{P}^r_{Kor} = \left(-p + \alpha\Delta\rho + \beta(\nabla\rho)^2\right)\mathbf{I} + \delta\nabla\rho\circ\nabla\rho + \gamma\nabla^2\rho$$

Origin of quantum potential – weakly nonlocal statistics:

*Unique under physically reasonable conditions.*

– Extensivity (mean, density)

– Isotropy

$$s(\rho, \nabla\rho) = \hat{s}(\rho, (\nabla\rho)^2)$$

– Additivity

$$s(\rho_1\rho_2, D(\rho_1\rho_2)) = s(\rho_1, \nabla\rho_1) + s(\rho_2, \nabla\rho_2)$$

$$\widehat{s}(\rho,(\nabla\rho)^2)=\nu\frac{(\nabla\rho)^2}{\rho^2}+k\ln\rho$$

Fisher

Boltzmann-Gibbs-Shannon

Extreme Physical Information (EPI) principle (Frieden, 1998)

– Mass-scale invariance (particle interpretation)

$$s(\lambda\rho,\lambda\nabla\rho)=s(\rho,\nabla\rho)$$

# Two component weakly nonlocal fluid

$$\rho = \nu_1 \gamma_1 + \nu_2 \gamma_2 = \nu\gamma$$

$\gamma$ density of the solid component
$\nu$ volume distribution function

$$\dot{\gamma} = 0$$
$$\dot{\nu} + \nu \nabla \cdot \mathbf{v} = 0$$
$$\nu\gamma\dot{\mathbf{v}} + \nabla \cdot \mathbf{P}(\mathrm{C}) = \mathbf{0}$$
$$\dot{s}(\mathrm{C}) + \nabla \cdot \mathbf{j}_s(\mathrm{C}) \geq 0$$

$(\gamma, \nu, \mathbf{v})$ basic state

$$C = (\gamma, \nabla\gamma, \nu, \nabla\nu, \mathbf{v}, \nabla\mathbf{v})$$

constitutive state

$s(\mathrm{C}), \mathbf{j}_s(\mathrm{C}), \mathbf{P}(\mathrm{C})$

constitutive functions

Constraints: $(1), \nabla(1), (2), \nabla(2), (3)$

Liu equations

$$\rho\partial_{\gamma} s = \Gamma_1,$$
$$\rho\partial_{\nabla\gamma} s = \Gamma_2,$$
$$\rho\partial_{\nu} s = \Gamma_3,$$
$$\rho\partial_{\nabla\nu} s = \Gamma_4,$$
$$\rho\partial_{\mathbf{v}} s = \Gamma_5,$$
$$\rho\partial_{\nabla\mathbf{v}} s = \mathbf{0},$$
$$(\partial_{\nabla\nu}\mathbf{j}_s - \Gamma_5\partial_{\nabla\nu}\mathbf{P})^s = \mathbf{0},$$
$$(\partial_{\nabla\lambda}\mathbf{j}_s - \Gamma_5\cdot\partial_{\nabla\gamma}\mathbf{P})^s = \mathbf{0},$$
$$(\partial_{\nabla\mathbf{v}}\mathbf{j}_s - \Gamma_4\nu\mathbf{I} - \Gamma_5\cdot\partial_{\nabla\mathbf{v}}\mathbf{P})^s = \mathbf{0}.$$

$$\partial_{\nu\nabla\nu} s = \mathbf{0},$$
$$\partial_{\nu\nabla\gamma} s = \mathbf{0},$$
$$\partial_{\nabla\nu\nabla\nu} s = \mathbf{0},$$
$$\partial_{\nabla\nu\nabla\gamma} s = \mathbf{0}.$$

isotropic, second order

Solution:

$$\boxed{s(\nu, \nabla\nu, \lambda, \nabla\gamma, \mathbf{v}) = s_e(\nu, \lambda) - m(\nu, \lambda)\,\frac{\mathbf{v}^2}{2} - \alpha(\nu, \lambda)\,\frac{(\nabla\gamma)^2}{2}}$$

$$\mathbf{j}_s(C) = -m(\nu, \lambda)\ \mathbf{v} \cdot \mathbf{P}(C) - \mathbf{j}_1(\nu, \lambda, \mathbf{v}).$$

$$-\nabla(m\mathbf{v}) : \mathbf{P} + \rho(\alpha\nabla\gamma \circ \nabla\gamma - \nu\partial_\nu s\mathbf{I}) : \nabla\mathbf{v} \geq 0$$

Simplification:

$$m = 1, \quad \mathbf{j}_1(\nu, \lambda, \mathbf{v}) = \mathbf{0}, \quad \partial_\nu s_e = -\frac{p}{\rho^2}.$$

Entropy inequality:

$$-\left(\mathbf{P}-\underbrace{\left(p+\rho\partial_{\nu}\alpha\frac{(\nabla\gamma)^{2}}{2}\right)\mathbf{I}-\rho\alpha\nabla\gamma\circ\nabla\gamma}_{\mathbf{P}^{r}}\right):\nabla\mathbf{v}\geq 0$$

Coulomb-Mohr

$$\mathbf{P}-\mathbf{P}^{r}=\mathbf{P}^{v}=-\mathbf{L}\nabla\mathbf{v}$$

isotropy: Navier-Stokes like + ...

# Properties

1 Other models:

a) Goodman-Cowin

$$\mathbf{P}^{r} = (p + \alpha(\nabla v)^2 + 2\alpha v \Delta v)\mathbf{I} - 2\alpha \nabla v \circ \nabla v$$

$$\gamma \ddot{v} - \nabla \cdot \mathbf{h} = \sigma_{\dot{v}}$$ configurational force balance

b) Navier-Stokes type: $+\nabla\rho$ somewhere

2 Coulomb-Mohr

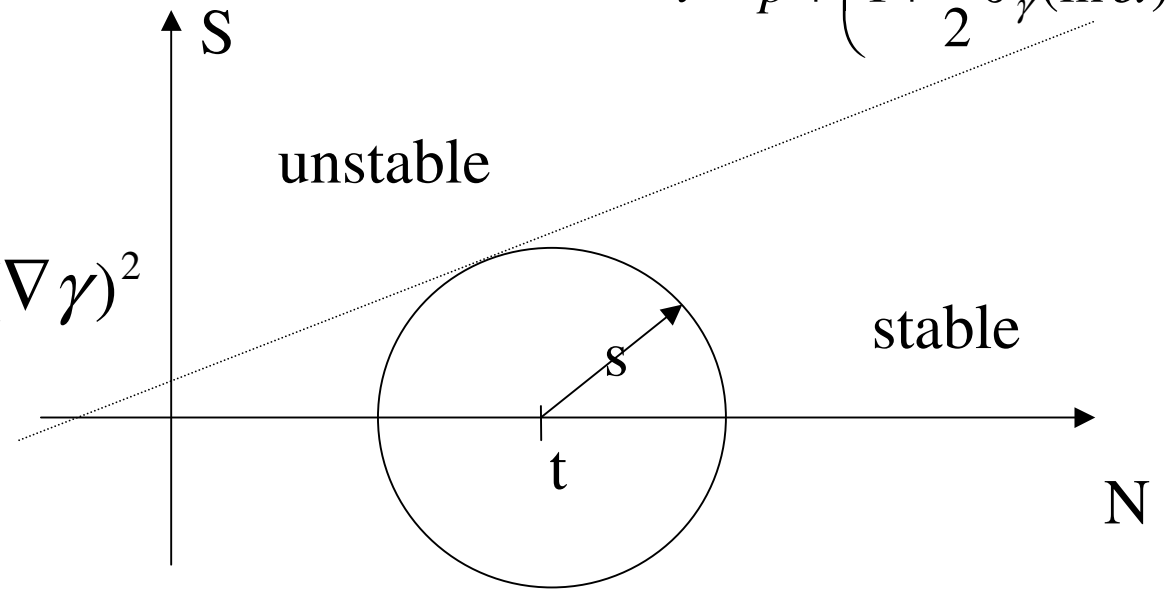
$$\mathbf{N} := \mathbf{n} \cdot \mathbf{P}^{\mathrm{r}} \cdot \mathbf{n} \qquad \mathbf{S} := \mathbf{P}^{\mathrm{r}} - \mathbf{N}$$

$$\mathbf{S}^2 + (\mathbf{N} - \mathbf{t})^2 = \mathbf{s}^2$$

$$s = \rho\alpha(\nabla\gamma)^2$$

$$t = p + \rho\left(\frac{\partial_\nu \alpha}{2} - \alpha\right)(\nabla\gamma)^2$$

$$t = p + \left(1 + \frac{1}{2}\partial_\gamma(\ln\alpha)\right)s$$



# Conclusions

- Weakly nonlocal statistical physics
- Universality (Second Law – super-principle)
  - independent of interpretation
  - independent of micro details

    phenomenological background behind any statistical-kinetic theory (Kaniadakis - kinetic, Frieden-Plastino - maxent)

- Method - more theories/models
- Material stability

Thermodynamics = theory of material stability

e.g. phase transitions (gradient systems?)

What about quantum mechanics?

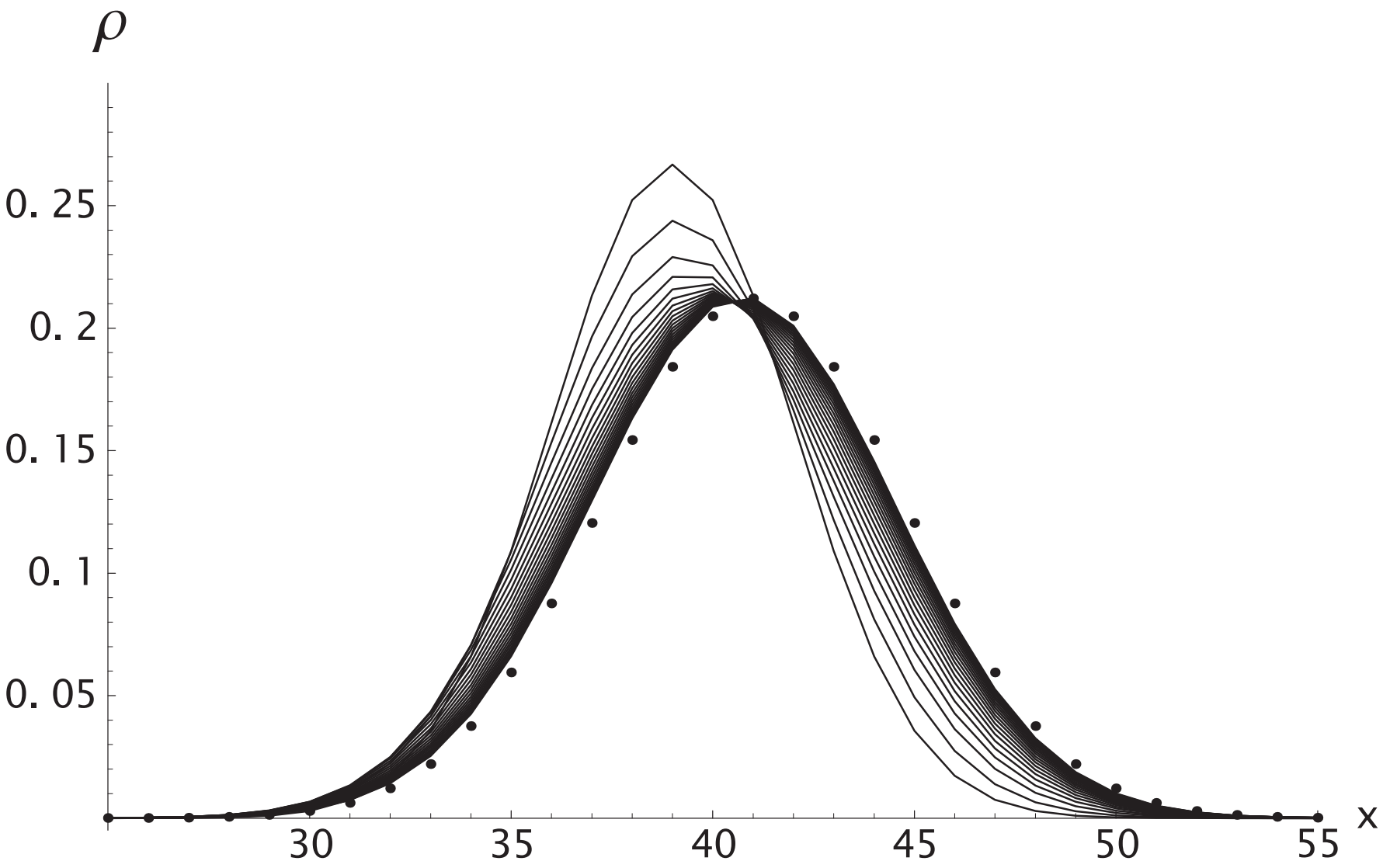
- There is a meaning of dissipation.
- There is a family of equilibrium (stationary) solutions.

$$\mathbf{v} = \mathbf{0} \qquad U_{SchM} + U = E = const.$$

- There is a thermodynamic Ljapunov function:

$$L(\rho, \mathbf{v}) = \int \rho \left( \frac{\mathbf{v}^2}{2} + \frac{1}{2} \left( \frac{\nabla \rho}{2\rho} \right)^2 + U - E \right) dV$$

semidefinite in a gradient (Soboljev ?) space


ρ
0. 25
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x