Thermodynamics of relativistic fluids

P. Ván

Department of Theoretical Physics
Research Institute of Particle and Nuclear Physics,
Budapest, Hungary

– Introduction
  –Thermodynamics and stability
– Non-relativistic fluids
– Stability paradox of dissipative relativistic fluids
– About the temperature of moving bodies
– Conclusions

Internal energy:

$$\varepsilon = \sqrt{e^2 - \mathbf{q}^2}$$

common work with T. S. Bíró and E. Molnár

Introduction – role of the Second Law:

Non-equilibrium thermodynamics:

basic variables
evolution equations (basic balances) $+$ Second Law

$\longrightarrow$ Stability of homogeneous equilibrium

$$\boxed{\text{Entropy} \sim \text{Lyapunov function}}$$

Homogeneous systems (equilibrium thermodynamics):

dynamic reinterpretation – ordinary differential equations
clear, mathematically strict

- Finite time thermodynamics ….
- Matolcsi T.: Ordinary thermodynamics (Academic Publishers, 2005)

Continuum systems

partial differential equations – Lyapunov theorem is more technical

⟶ Linear stability (of homogeneous equilibrium)

Example: non-relativistic fluid mechanics
local equilibrium, Fourier-Navier-Stokes

$$\dot{n} + n\partial_i v^i = 0,$$
$$\dot{\varepsilon} + \varepsilon\partial_i v^i + \partial_i q^i + P^{ij}\partial_i v_j = 0,$$
$$\dot{p}^i + p^i\partial_j v^j + \partial_j P^{ij} = 0^i.$$

n	*particle number density*
v^i	*relative (3-)velocity*
e	*internal energy density*
q^i	*internal energy (heat) flux*
P^{ij}	*pressure*
p^i	*momentum density*

Thermodynamics

$$d\varepsilon = Tds + \mu dn, \qquad p = Ts + \mu n - \varepsilon, \quad \varepsilon = e - \frac{\rho v^2}{2}.$$

$$\dot{s}(\varepsilon, n) + s\partial_i v^i + \partial_i J^i = \frac{1}{T}\dot{\varepsilon} - \frac{\mu}{T}\dot{n} + s\partial_i v^i + \partial_i \frac{q^i}{T} = \ldots =$$

$$\boxed{q^i\partial_i \frac{1}{T} - \frac{1}{T}\left(P^{ij} - \underbrace{(Ts + \mu n - \varepsilon)}_{p}\delta^{ij}\right)\partial_i v_j \geq 0}$$

$$\dot{n} + n\partial_i v^i = 0,$$
$$\dot{\varepsilon} + \varepsilon\partial_i v^i + \partial_i q^i + P^{ij}\partial_i v_j = 0,$$
$$\dot{p}^i + p^i\partial_j v^j + \partial_j P^{ij} = 0^i,$$

Fourier-Navier-Stokes

$$\boxed{\begin{aligned} q^i &= -\lambda\partial^i T, \\ \Pi^{ij} &= -\xi\partial_k v^k \delta^{ij} - 2\eta\left\langle\partial_i v^j\right\rangle. \end{aligned}}$$

linear constitutive relations,
<> is symmetric, traceless part

Equilibrium:

$$n(x_i, t) = const., \quad \varepsilon(x_i, t) = const., \quad v^i(x_i, t) = const.$$

Linearization, …, Routh-Hurwitz criteria:

$$\lambda > 0, \quad \eta > 0, \quad \xi > 0,$$
$$\partial_\varepsilon T > 0,$$

Thermodynamic stability
(concave entropy)

$$\underbrace{(\varepsilon + p)\partial_\varepsilon p + n\partial_n p > 0}_{\text{Hydrodynamic stability}} \quad \Leftrightarrow \quad \underbrace{\partial_\varepsilon T \partial_n \frac{\mu}{T} - \partial_n T \partial_\varepsilon \frac{\mu}{T}}_{Det(\partial^2 s)} > 0$$

Dissipative relativistic fluids

	Nonrelativistic	Relativistic
Local equilibrium (1st order)	Fourier+Navier-Stokes	~~Eckart~~ (1940)
Beyond local equilibrium (2^{nd} order)	Cattaneo-Vernotte, gen. Navier-Stokes	Israel-Stewart (1969-72), Müller-Ruggieri, Öttinger, Carter, etc.

Eckart:

$$S^a(T^{ab}, N^a) = s(e,n)u^a + \frac{q^a}{T}$$

Israel-Stewart:

$$S^a(T^{ab}, N^a) = \left(\underline{s(e,\, n)} - \frac{\beta_0}{2}\pi^2 - \frac{\beta_1}{2} q_b q^b - \frac{\beta_2}{2}\Pi^{bc}\Pi_{bc} \right) u^a +$$

$$+ \underline{\frac{1}{T}}\left(q^a + \alpha_0 \pi q^a + \alpha_1 \Pi^{ab} q_b \right)$$

Special relativistic fluids (Eckart):

$$T^{ab} = eu^a u^b + q^a u^b + q^b u^a + P^{ab},$$

$$N^a = nu^a + j^a.$$

energy-momentum density
particle density vector

$$q^a u_a = j^a u_a = 0, \quad P^{ba} u_a = P^{ab} u_a = 0^b$$

General representations by local rest frame quantities.

$$\boxed{\partial_a S^a = \dot{s}(e,n) + s\partial_a u^a + \partial_a J^a \geq 0}$$

$$\dot{e} = u^a \partial_a e$$

$$J^a = \frac{q^a}{T}$$

$$u^a \partial_b T^{ab} = \dot{e} + e\partial_a u^a + \partial_a q^a + u_a \dot{q}^a + \ldots$$

$$\partial_b N^b = \dot{n} + n\partial_a u^a + \partial_a j^a$$

$$\boxed{q^i \partial_i \frac{1}{T} - \frac{1}{T}\left(P^{ij} - p\delta^{ij}\right)\partial_i v_j \geq 0}$$

$$\sigma_s = j^a \partial_a \frac{\mu}{T} - \frac{1}{T}(P^{ab} - p\delta^{ab})\partial_b u_a - \frac{q^a}{T^2}(\partial_a T \boxed{- T\dot{u}_a}) \geq 0$$

Eckart term

Stability of homogeneous equilibrium

Eckart theory:

instable –

due to heat conduction

$$\tau \leq \frac{\kappa T}{(\rho c^2 + p)c^2} \approx 10^{-34} s$$

Israel-Stewart theory:

$\rightarrow$ stability is conditional:

complicated conditions

$$\beta_1 (p + e) > 1$$

$\rightarrow$ relaxation to the first order theory? (Geroch 1995, Lindblom 1995)

Second Law as a constrained inequality (Liu procedure):

1) $$s(e,u^a) = s(e,q^a(e,u^a))$$

2) $$e\frac{\partial s}{\partial q^a} = q_a \frac{\partial s}{\partial e} \Rightarrow$$

$$s(e,q^a) = \hat{s}(e^2 - \mathbf{q}^2) = \tilde{s}\left(\sqrt{e^2 - \mathbf{q}^2}\right)$$

Ván: under publication in JMMS, (arXiv:07121437)

Modified relativistic irreversible thermodynamics:

Internal energy: $$\boxed{\varepsilon = \sqrt{e^2 - \mathbf{q}^2} = \sqrt{\varepsilon^a \varepsilon_a} = \sqrt{u_b T^{ba} T_{ac} u^c}}$$

$$\partial_a S^a = \dot{s}(\varepsilon, n) + s\partial_a u^a + \partial_a J^a \geq 0$$

$$J^a = \frac{q^a}{T}$$

$$u^a \partial_b T^{ab} = \dot{e} + e\partial_a u^a + \partial_a q^a + u_a \dot{q}^a + u_a \partial_b P^{ab} = 0$$

$$\partial_b N^b = \dot{n} + n\partial_a u^a + \partial_a \boldsymbol{\upsilon}^a$$

$$\sigma_s = \upsilon^a \partial_a \frac{\mu}{T} - \frac{1}{T}\left(P^{ab} - p\delta^{ab}\right)\partial_b u_a - \frac{q^a}{T^2}\left(\partial_a T + \boxed{T\dot{u}_a} \boxed{+T\frac{\dot{q}_a}{e}}\right) \geq 0$$

Eckart term

Dissipative hydrodynamics

$$
\begin{aligned}
\partial_a N^a &= \dot{n} + n\partial_a u^a + \partial_a j^a = 0, \\
u^a \partial_b T^{ab} &= \dot{e} + (e+p)\partial_a u^a + \partial_a q^a + q^a \dot{u}_a - \Pi^{ab}\partial_b u_a = 0, \\
\Delta^a_c \partial_b T^{cb} &= (e+p)\dot{u}^a + q^a \partial_b u^b + q^b \partial_b u^a + \Delta^a_c(\dot{q}^c + \partial_b \Pi^{cb}) = 0^a, \\
q^a &= -\lambda \Delta^{ac}\left(\partial_c T + T\dot{u}_c + \boxed{T\frac{\dot{q}^a}{e}}\right), \\
\nu^a &= -\varsigma \Delta^{ac}\partial_c \frac{\mu}{T}, \\
\Pi^a_a &= P^a_a - p = -\xi \partial_c u^c, \\
\Pi^a_b &= -2\eta < \partial_b u^a >.
\end{aligned}
$$

$< \, >$ symmetric traceless spacelike part

$\Rightarrow$ linear stability of homogeneous equilibrium

CONDITION: thermodynamic stablity

Thermodynamics:

$$de - \frac{q^a}{e} dq_a = Tds + \mu dn \Rightarrow s\left(\sqrt{e^2 + q^a q_a}, n\right) = \hat{s}\left(e, q^a, n\right)$$

Temperatures and other intensives are doubled:

$$\frac{\partial s}{\partial \varepsilon} = \frac{1}{\Theta}; \quad \frac{\partial \hat{s}}{\partial e} = \frac{1}{T} \quad \Rightarrow \quad eT = \varepsilon \Theta, e\mu = \varepsilon \mathrm{M}$$

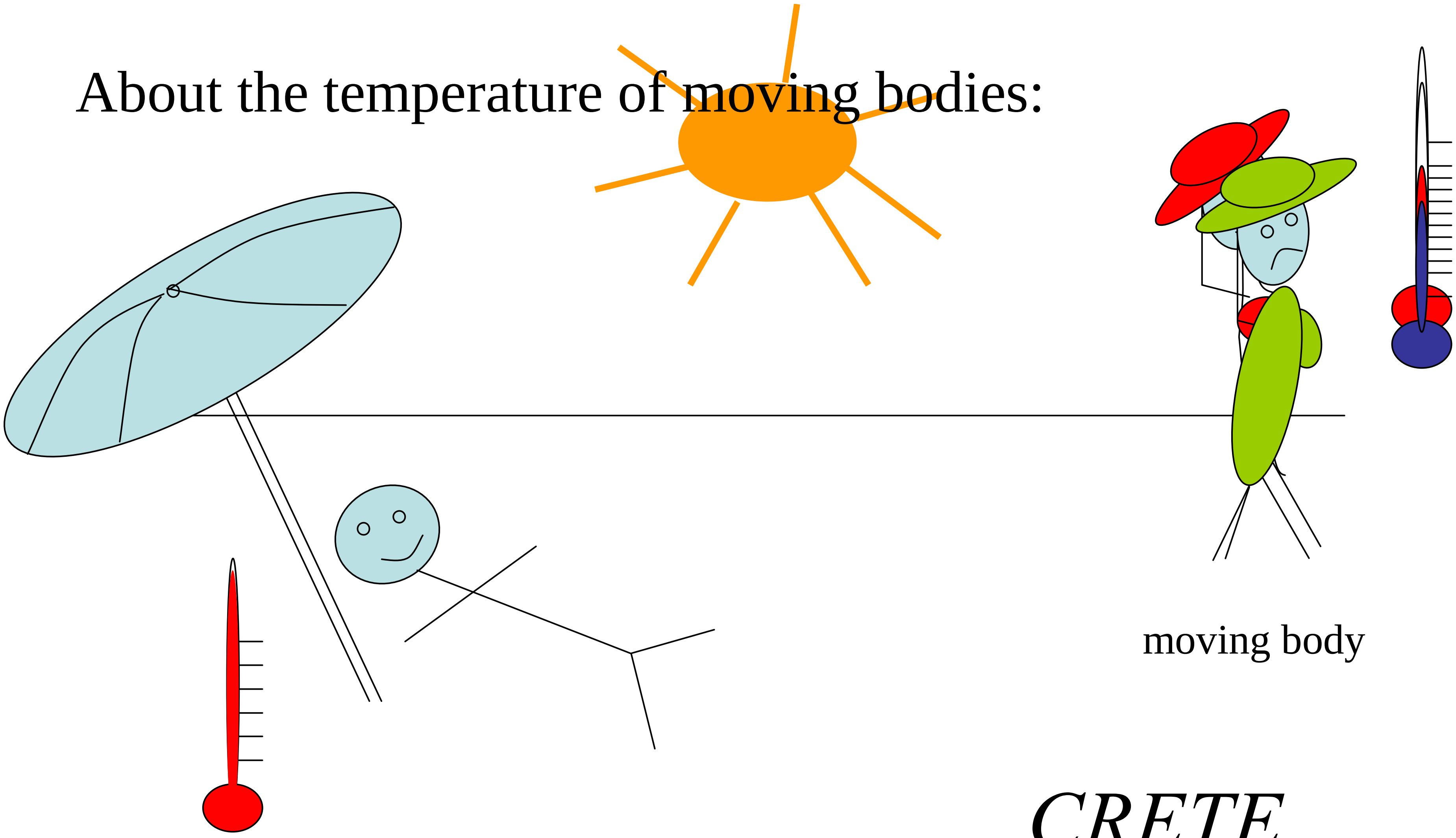
Different roles:

Equations of state: Θ, M

Constitutive functions: T, μ

$$q^a = -\lambda \Delta^{ac}\left(\partial_c T + T\dot{u}_c + T\frac{\dot{q}^a}{e}\right)$$

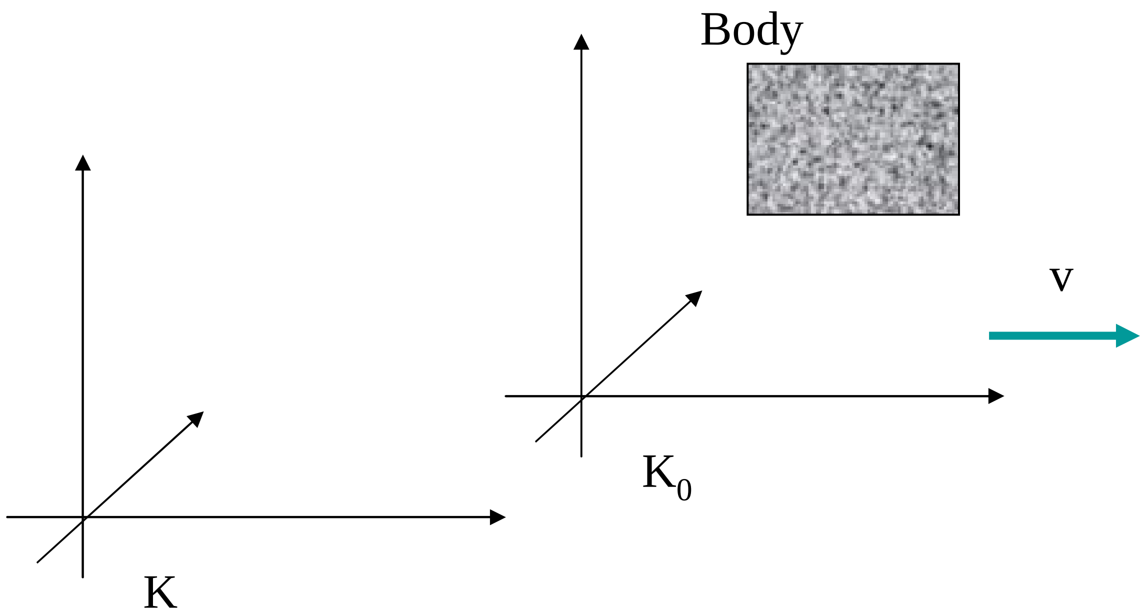
About the temperature of moving bodies:



CRETE

ΣΦ 2008

About the temperature of moving bodies:



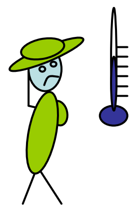
$$\gamma = \frac{1}{\sqrt{1-v^2}}$$

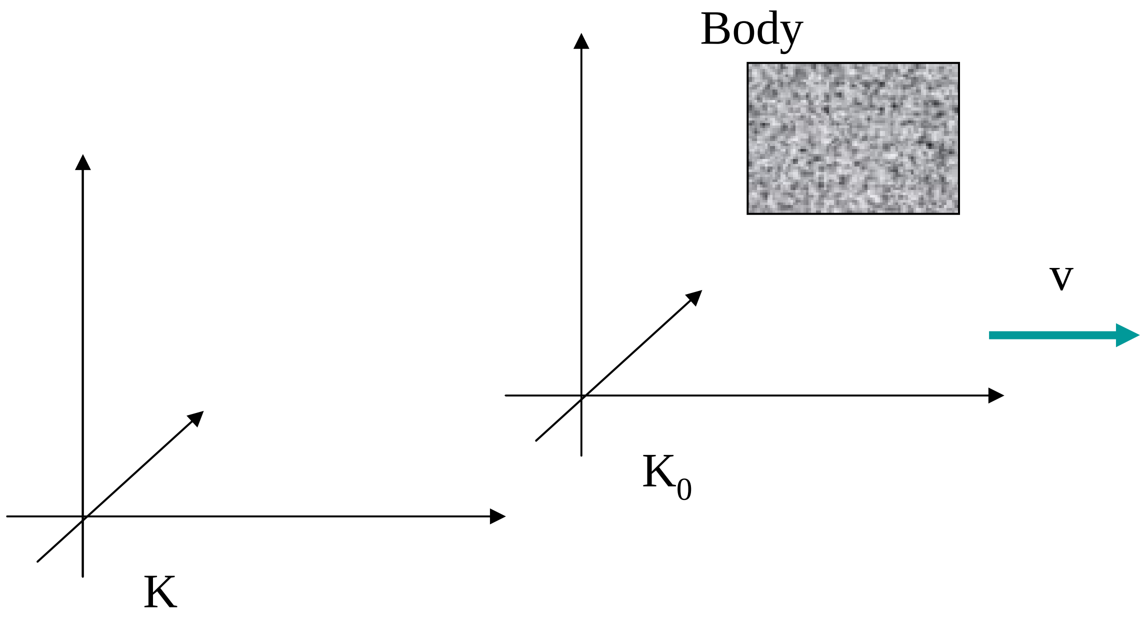
$$dE = TdS - pdV - \mathbf{v}d\mathbf{G}$$

translational work

Einstein-Planck: entropy is invariant, energy is vector

$$S = S_0, \quad V = \gamma^{-1} V_0, \quad E = \gamma E_0 \quad \Rightarrow \quad \boxed{T = \gamma^{-1} T_0}, \quad p = p_0$$





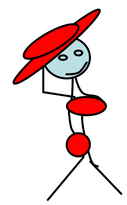
$$dE = TdS - pdV \qquad \gamma = \frac{1}{\sqrt{1-v^2}}$$

Ott - hydro: entropy is vector, energy-presssure are from a tensor

$$S = \gamma S_0, \quad E = \gamma^2 E_0, \quad p = \gamma^2 p_0 \quad \Rightarrow \quad \boxed{T = \gamma T_0}$$

Our:

$$de - \frac{q^a}{e} dq_a = Tds + \mu dn$$



Summary

– energy $\neq$ internal energy

$\rightarrow$ generic stability without extra conditions

- relativistic thermodynamics: there is no local equilibrium

- different temperatures in Fourier-law (equilibration) and in state functions out of local equilibrium.

- causality

/Ván and Bíró, EPJ, (2007), **155**, p201-212, (arXiv:0704.2039v2)/

- hyperbolic(-like) extensions, solutions

/Bíró, Molnár and Ván: under publication in PRC, (arXiv:0805.1061)/

Thank you for your attention!