Thermodynamics of relativistic fluids

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– Introduction
  – Thermodynamics and stability
– Non-relativistic fluids
– Stability paradox of dissipative relativistic fluids
– About the temperature of moving bodies
– Conclusions

common work with T. S. Bíró and E. Molnár

Internal energy:
\[ \epsilon = \sqrt{e^2 - q^2} \]
Introduction – role of the Second Law:

Non-equilibrium thermodynamics:

- basic variables
- evolution equations (basic balances)
- Second Law

→ Stability of homogeneous equilibrium

Entropy ~ Lyapunov function

Homogeneous systems (equilibrium thermodynamics):

- dynamic reinterpretation – ordinary differential equations
- clear, mathematically strict

- Finite time thermodynamics ….
Continuum systems

partial differential equations – Lyapunov theorem is more technical

Linear stability (of homogeneous equilibrium)

Example: non-relativistic fluid mechanics

local equilibrium, Fourier-Navier-Stokes

\[\dot{n} + n \partial_i v^i = 0,\]

\[\dot{\varepsilon} + \varepsilon \partial_i v^i + \partial_i q^i + P_{ij} \partial_j v = 0,\]

\[\dot{P}^i + P^i \partial_j v^j + \partial_j P_{ij} = 0.\]

Thermodynamics

\[d\varepsilon = T ds + \mu dn,\]

\[p = Ts + \mu n - \varepsilon,\]

\[\varepsilon = e - \frac{\rho v^2}{2}.\]

\[s(\varepsilon, n) + s \partial_i v^i + \partial_i J^i = \frac{1}{T} \dot{\varepsilon} - \frac{\mu}{T} \dot{n} + s \partial_i v^i + \partial_i \frac{q^i}{T} = \ldots =\]

\[q^i \partial_i \frac{1}{T} - \frac{1}{T} \left( P_{ij} - (Ts + \mu n - \varepsilon) \delta_{ij} \right) \partial_i v^j \geq 0\]
\[ \dot{n} + n \partial_i v^i = 0, \]
\[ \dot{\varepsilon} + \varepsilon \partial_i v^i + \partial_i q^i + P^{ij} \partial_j v_j = 0, \]
\[ \dot{p}^i + p^i \partial_j v^j + \partial_j P^{ij} = 0^i, \]
\[ q^i = -\lambda \partial^j T, \]
\[ \Pi^{ij} = -\xi \partial_k v^k \delta^{ij} - 2\eta \langle \partial_i v^j \rangle. \]

**Equilibrium:**

\[ n(x_i, t) = \text{const.}, \quad \varepsilon(x_i, t) = \text{const.}, \quad v^i(x_i, t) = \text{const}. \]

**Linearization, …, Routh-Hurwitz criteria:**

\[ \lambda > 0, \quad \eta > 0, \quad \xi > 0, \]
\[ \partial \varepsilon T > 0, \]
\[ (\varepsilon + p) \partial \varepsilon p + n \partial_n p > 0 \quad \Leftrightarrow \quad \partial \varepsilon T \partial_n \frac{\mu}{T} - \partial_n T \partial \varepsilon \frac{\mu}{T} > 0 \]

Thermodynamic stability (concave entropy)

hydrodynamic stability

\[ \text{Det}(\partial^2 s) \]
Dissipative relativistic fluids

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Eckart:

\[ S^a (T^{ab}, N^a) = s(e, n)u^a + \frac{q^a}{T} \]

Israel-Stewart:

\[ S^a (T^{ab}, N^a) = \left( s(e, n) - \frac{\beta_0}{2} \pi^2 - \frac{\beta_1}{2} q_b q^b - \frac{\beta_2}{2} \Pi^{bc} \Pi_{bc} \right) u^a + \]

\[ + \frac{1}{T} \left( q^a + \alpha_0 \pi q^a + \alpha_1 \Pi^{ab} q_b \right) \]
Special relativistic fluids (Eckart):

\[ T^{ab} = eu^a u^b + q^a u^b + q u^a + P^{ab}, \]
\[ N^a = nu^a + j^a. \]

\[ q^a u_a = j^a u_a = 0, \quad P^{ba} u_a = P^{ab} u_a = 0^b \]

General representations by local rest frame quantities.

\[ \partial_a S^a = \dot{s}(e, n) + s \partial_a u^a + \partial_a J^a \geq 0 \]
\[ \dot{e} = u^a \partial_a e \]
\[ J^a = \frac{q^a}{T} \]

\[ u^a \partial_b T^{ab} = \dot{e} + e \partial_a u^a + \partial_a q^a + u^a \dot{q}^a + \partial_b N^b = \dot{n} + n \partial_a u^a + \partial_a j^a \]

\[ q^i \partial_i \frac{1}{T} - \frac{1}{T} \left( P^{ij} - p \delta^{ij} \right) \partial_i v_j \geq 0 \]

\[ \sigma_s = j^a \partial_a \frac{\mu}{T} - \frac{1}{T} \left( P^{ab} - p \delta^{ab} \right) \partial_b u_a - \frac{q^a}{T^2} \left( \partial_a T - Tu_a \right) \geq 0 \]

Eckart term
Stability of homogeneous equilibrium

Eckart theory:

- **instable** –
  - due to heat conduction

\[
\tau \leq \frac{\kappa T}{(\rho c^2 + p)c^2} \approx 10^{-34} \text{ s}
\]

Israel-Stewart theory:

→ stability is conditional: \( \beta_1 (p + e) > 1 \)

complicated conditions

→ relaxation to the first order theory? (Geroch 1995, Lindblom 1995)
Second Law as a constrained inequality (Liu procedure):

1) \[ s(e, u^a) = s(e, q^a(e, u^a)) \]

2) \[ e \frac{\partial s}{\partial q^a} = q^a \frac{\partial s}{\partial e} \Rightarrow \]

\[ s(e, q^a) = \tilde{s}(e^2 - q^2) = \tilde{s} \left( \sqrt{e^2 - q^2} \right) \]

Ván: under publication in JMMS, (arXiv:07121437)
Modified relativistic irreversible thermodynamics:

Internal energy: 

\[ \varepsilon = \sqrt{e^2 - q^2} = \sqrt{\varepsilon^a \varepsilon_a} = \sqrt{u_b T^{ba} T_{ac} u^c} \]

\[ \partial_a S^a = s(\varepsilon, n) + s \partial_a u^a + \partial_a J^a \geq 0 \]

\[ J^a = \frac{q^a}{T} \]

\[ u^a \partial_b T^{ab} = \dot{e} + e \partial_a u^a + \partial_a q^a + u_a \dot{q}^a + u_a \partial_b P^{ab} = 0 \]

\[ \partial_b N^b = \dot{\mathbf{n}} + n \partial_a u^a + \partial_a \mathbf{v}^a \]

\[ \sigma_s = \nu^a \partial_a \frac{\mu}{T} - \frac{1}{T} \left( P^{ab} - p \delta^{ab} \right) \partial_b u_a - \frac{q^a}{T^2} \left( \partial_a T + T \frac{\dot{u}_a}{e} + T \frac{\dot{q}_a}{e} \right) \geq 0 \]

Eckart term
Dissipative hydrodynamics

\[ \partial_a N^a = \dot{n} + n \partial_a u^a + \partial_a j^a = 0, \]
\[ u^a \partial_b T^{ab} = \dot{e} + (e + p) \partial_a u^a + \partial_a q^a + q^a \dot{u}_a - \Pi^{ab} \partial_b u_a = 0, \]
\[ \Delta_c \partial_b T^{cb} = (e + p) \dot{u}^a + q^a \partial_b u^b + q^b \partial_b u^a + \Delta_c (\dot{q}^c + \partial_b \Pi^{cb}) = 0^a, \]
\[ q^a = -\lambda \Delta^{ac} \left( \partial_c T + T \dot{u}_c + T \frac{\dot{q}^a}{e} \right), \]
\[ \nu^a = -\zeta \Delta^{ac} \frac{\mu}{T}, \]
\[ \Pi^a_a = P^a_a - p = -\xi \partial_c u^c, \]
\[ \Pi^a_b = -2\eta \langle \partial_b u^a \rangle. \]

\[ \Rightarrow \text{linear stability of homogeneous equilibrium} \]

CONDITION: thermodynamic stability
Thermodynamics:

\[ de - \frac{q^a}{e} dq_a = Tds + \mu dn \implies s\left(\sqrt{e^2 + q^a q_a}, n\right) = \hat{s}(e, q^a, n) \]

Temperatures and other intensives are doubled:

\[
\frac{\partial s}{\partial \varepsilon} = \frac{1}{\Theta} ; \quad \frac{\partial \hat{s}}{\partial e} = \frac{1}{T} \implies eT = \varepsilon \Theta, \ e\mu = \varepsilon M
\]

Different roles:

Equations of state: \( \Theta, M \)
Constitutive functions: \( T, \mu \)

\[ q^a = -\lambda \Delta^{ac} \left( \partial_c T + T \dot{u}_c + T \frac{\dot{q}^a}{e} \right) \]
About the temperature of moving bodies:

CRETE

ΣΦ 2008
About the temperature of moving bodies:

\[ dE = TdS - pdV - \mathbf{v}d\mathbf{G} \]

translational work

**Einstein-Planck:** entropy is invariant, energy is vector

\[ S = S_0, \quad V = \gamma^{-1}V_0, \quad E = \gamma E_0 \quad \Rightarrow \quad T = \gamma^{-1}T_0, \quad p = p_0 \]
\[ dE = TdS - pdV \]

\[ \gamma = \frac{1}{\sqrt{1 - v^2}} \]

Ott - hydro: entropy is vector, energy-pressure are from a tensor

\[ S = \gamma S_0, \quad E = \gamma^2 E_0, \quad p = \gamma^2 p_0 \quad \Rightarrow \quad T = \gamma T_0 \]

Our:

\[ de - \frac{q^a}{e} dq_a = Tds + \mu dn \]
Summary

- energy ≠ internal energy
  → generic stability without extra conditions

- relativistic thermodynamics: there is no local equilibrium

- different temperatures in Fourier-law (equilibration) and in state functions out of local equilibrium.

- causality

- hyperbolic(-like) extensions, solutions
  /Bíró, Molnár and Ván: under publication in PRC, (arXiv:0805.1061)/
Thank you for your attention!