Hydrodynamics, stability and temperature

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- Motivation
  - Problems with second order theories
  - Thermodynamics, fluids and stability
- Generic stability of relativistic dissipative fluids
- Temperature of moving bodies
- Summary

Internal energy:

$$\varepsilon = \sqrt{e^2 - \mathbf{q}^2}$$

Dissipative relativistic fluids

	Nonrelativistic	Relativistic
Local equilibrium (1st order)	Fourier+Navier-Stokes	Eckart (1940), Tsumura-Kunihiro (2008)
Beyond local equilibrium (2^{nd} order)	Cattaneo-Vernotte, gen. Navier-Stokes	Israel-Stewart (1969-72), Pavón, Müller-Ruggieri, Geroch, Öttinger, Carter, conformal, etc.

Eckart:

$$S^a(T^{ab}, N^a) = s(e,n)u^a + \frac{q^a}{T}$$

Extended (Israel–Stewart – Pavón–Jou–Casas-Vázquez):

$$S^a(T^{ab}, N^a) = \left(\underline{s(e,\, n)} - \frac{\beta_0}{2T}\Pi^2 - \frac{\beta_1}{2T} q_b q^b - \frac{\beta_2}{2T}\pi^{bc}\pi_{bc} \right) u^a +$$

$$+ \underline{\frac{1}{T}} \left(q^a + \alpha_0 \Pi q^a + \alpha_1 \pi^{ab} q_b \quad \right)$$

(+ order estimates)

Remarks on causality and stability:

Symmetric hyperbolic equations ~ causality

$$\partial_t T - \kappa \partial_{xx} T = 0,$$

$$\gamma(\partial_{\tilde{t}} - v\partial_{\tilde{x}})T - \kappa\left(\partial_{\tilde{x}\tilde{x}} - 2\frac{v}{c^2}\partial_{\tilde{x}\tilde{t}} + \frac{v^2}{c^4}\partial_{\tilde{t}\tilde{t}}\right)T = 0$$

– The extended theories are *not* proved to be symmetric hyperbolic.

– In Israel-Stewart theory the symmetric hyperbolicity *conditions* of the *perturbation* equations follow from the stability conditions.

– Parabolic theories cannot be excluded – speed of the validity range can be small. Moreover, they can be extended later.

Stability of the homogeneous equilibrium (generic stability) is required.

– Fourier-Navier-Stokes limit. Relaxation to the (unstable) first order theory? (Geroch 1995, Lindblom 1995)

Fourier-Navier-Stokes

$$\dot{n} + n\partial_i v^i = 0,$$
$$\dot{\varepsilon} + \varepsilon\partial_i v^i + \partial_i q^i + P^{ij}\partial_i v_j = 0,$$
$$\dot{k}^i + k^i\partial_j v^j + \partial_j P^{ij} = 0^i,$$

$$q^i = -\lambda\partial^i T,$$
$$\Pi^{ij} = -\xi\partial_k v^k \delta^{ij} - 2\eta\langle\partial_i v^j\rangle.$$

$$q^i\partial_i \frac{1}{T} - \frac{1}{T}\left(P^{ij} - \underbrace{(Ts + \mu n - \varepsilon)}_{p}\delta^{ij}\right)\partial_i v_j \geq 0$$

Isotropic linear constitutive relations,
<> is symmetric, traceless part

Equilibrium:

$$n(x_i, t) = const., \quad \varepsilon(x_i, t) = const., \quad v^i(x_i, t) = const.$$

Linearization, …, Routh-Hurwitz criteria:

$$\lambda > 0, \quad \eta > 0, \quad \xi > 0,$$
$$\partial_\varepsilon T > 0,$$

Thermodynamic stability (concave entropy)

$$\underbrace{(\varepsilon + p)\partial_\varepsilon p + n\partial_n p > 0}_{\text{Hydrodynamic stability}} \quad \Leftrightarrow \quad \underbrace{\partial_\varepsilon T\partial_n \frac{\mu}{T} - \partial_n T\partial_\varepsilon \frac{\mu}{T} > 0}_{-T^2 Det(\partial^2 s)}$$

Remarks on stability and Second Law:

Non-equilibrium thermodynamics:

basic variables
evolution equations (basic balances) } + Second Law

⟶ Stability of homogeneous equilibrium

Entropy ~ Lyapunov function

<u>Homogeneous systems</u> (equilibrium thermodynamics):

dynamic reinterpretation – ordinary differential equations

clear, mathematically strict

See e.g. Matolcsi, T.: Ordinary thermodynamics, Academic Publishers, 2005

<u>Continuum systems</u> (irreversible thermodynamics):

partial differential equations – Lyapunov theorem is more technical

⟶ Linear stability (of homogeneous equilibrium)

Stability conditions of the Israel-Stewart theory

(Hiscock-Lindblom 1985)

$$\Omega_1 = \frac{1}{e+p}\frac{\partial e}{\partial p}\bigg|_{\frac{s}{n}} = \frac{T}{(e+p)\frac{\partial p}{\partial e}\Big|_n - n\frac{\partial p}{\partial n}\Big|_e} \geq 0,$$

$$\Omega_2 = \frac{1}{e+p}\frac{\partial e}{\partial (s/n)}\bigg|_p \frac{\partial p}{\partial (s/n)}\bigg|_{\frac{\mu}{nT}} = \ldots \geq 0,$$

$$\Omega_5 = \beta_0 \geq 0, \qquad \Omega_8 = \beta_2 \geq 0, \qquad \Omega_7 = \beta_1 - \frac{\alpha_1^2}{2\beta_2} \geq 0,$$

$$\Omega_4 = e + p - \frac{2\beta_2 + \beta_1 + 2\alpha_1}{2\beta_1\beta_2 - \alpha_1^2} \geq 0, \quad \Omega_6 = \beta_1 - \frac{\alpha_0^2}{\beta_0} - \frac{2\alpha_1^2}{3\beta_2} - \frac{1}{n^2 T}\frac{\partial T}{\partial (s/n)}\bigg|_n \geq 0,$$

$$\Omega_3 = (e+p)\left(1 - \frac{\partial p}{\partial e}\bigg|_{\frac{s}{n}}\right) - \frac{1}{\beta_0} - \frac{2}{3\beta_2} - \frac{K^2}{\Omega_6} \geq 0,$$

$$K = 1 + \frac{\alpha_0}{\beta_0} + \frac{2\alpha_1}{3\beta_2} - \frac{n}{T}\frac{\partial T}{\partial n}\bigg|_{s/n} \geq 0.$$

Special relativistic fluids (Eckart):

$$T^{ab} = eu^a u^b + q^a u^b + q^b u^a + P^{ab},$$ energy-momentum density

$$N^a = nu^a + j^a.$$ particle density vector

$$q^a u_a = j^a u_a = 0, \quad P^{ba} u_a = P^{ab} u_a = 0^b$$

q^a – momentum density or energy flux??

General representations by local rest frame quantities.

$$\partial_a S^a = \dot{s}(e,n) + s\partial_a u^a + \partial_a J^a \geq 0$$

$$\dot{e} = u^a \partial_a e$$

$$J^a = \frac{q^a}{T} - \frac{\mu}{T} j^a$$

$$u^a \partial_b T^{ab} = \dot{e} + e\partial_a u^a + \partial_a q^a + u_a \dot{q}^a$$

$$\partial_b N^b = \dot{n} + n\partial_a u^a + \partial_a j^a$$

$$-\frac{1}{T}\left(P^{ij} - p\delta^{ij}\right)\partial_i v_j + q^i \partial_i \frac{1}{T} \geq 0$$

$$\sigma_s = j^a \partial_a \frac{\mu}{T} - \frac{1}{T}(P^{ab} - p\delta^{ab})\partial_b u_a - \frac{q^a}{T^2}(\partial_a T - T\dot{u}_a) \geq 0$$

Eckart term

Modified relativistic irreversible thermodynamics:

Internal energy: $\varepsilon = \sqrt{e^2 - \mathbf{q}^2} = \sqrt{\varepsilon^a \varepsilon_a} = \sqrt{u_b T^{ba} T_{ac} u^c}$

$$\partial_a S^a = \dot{s}(\varepsilon, n) + s\partial_a u^a + \partial_a J^a \geq 0$$

$$J^a = \frac{q^a}{T} - \frac{\mu}{T} j^a$$

$$u^a \partial_b T^{ab} = \dot{e} + e\partial_a u^a + \partial_a q^a + u_a \dot{q}^a + u_a \partial_b P^{ab} = 0$$

$$\partial_b N^b = \dot{n} + n\partial_a u^a + \partial_a j^a$$

$$\sigma_s = j^a \partial_a \frac{\mu}{T} - \frac{1}{T}\left(P^{ab} - p\delta^{ab}\right)\partial_b u_a - \frac{q^a}{T^2}\left(\partial_a T + T\dot{u}_a + T\frac{\dot{q}_a}{e}\right) \geq 0$$

Eckart term

Ván and Bíró EPJ, (2008), **155**, 201. (arXiv:0704.2039v2)

Dissipative hydrodynamics

$$\partial_a N^a = \dot{n} + n\partial_a u^a + \partial_a j^a = 0,$$

$$u^a \partial_b T^{ab} = \dot{e} + (e+p)\partial_a u^a + \partial_a q^a + q^a \dot{u}_a - \Pi^{ab}\partial_b u_a = 0,$$

$$\Delta^a_c \partial_b T^{cb} = (e+p)\dot{u}^a + q^a \partial_b u^b + q^b \partial_b u^a + \Delta^a_c(\dot{q}^c + \partial_b \Pi^{cb}) = 0^a,$$

$$q^a = -\lambda \Delta^{ac}\left(\partial_c T + T\dot{u}_c + \boxed{T\frac{\dot{q}^a}{e}}\right),$$

$$\nu^a = -\varsigma \Delta^{ac}\partial_c \frac{\mu}{T},$$

$$\Pi^a_a = P^a_a - p = -\xi \partial_c u^c,$$

$$\Pi^a_b = -2\eta < \partial_b u^a > .$$

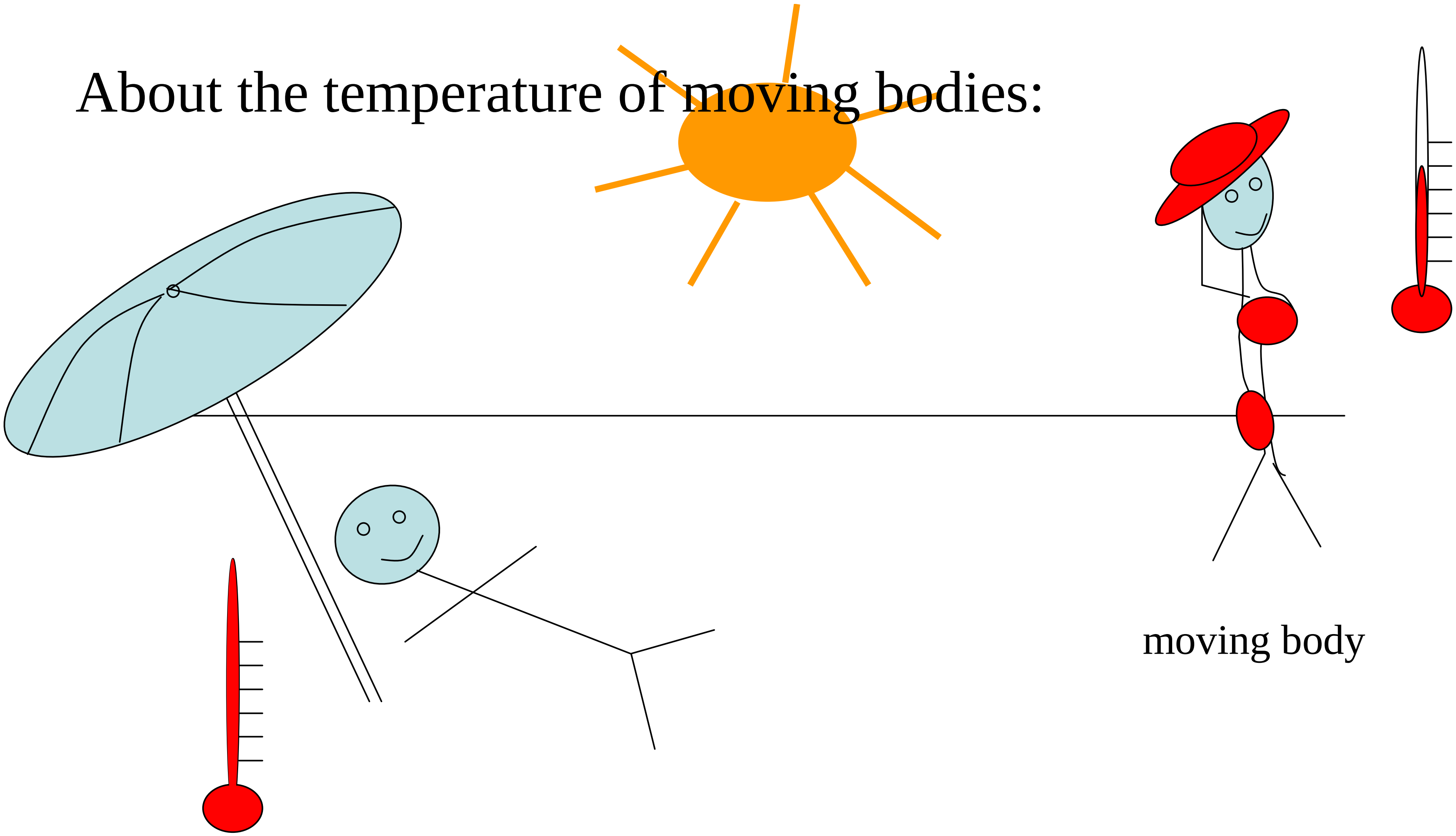
$< \ >$ symmetric traceless spacelike part

$\Rightarrow$ linear stability of homogeneous equilibrium

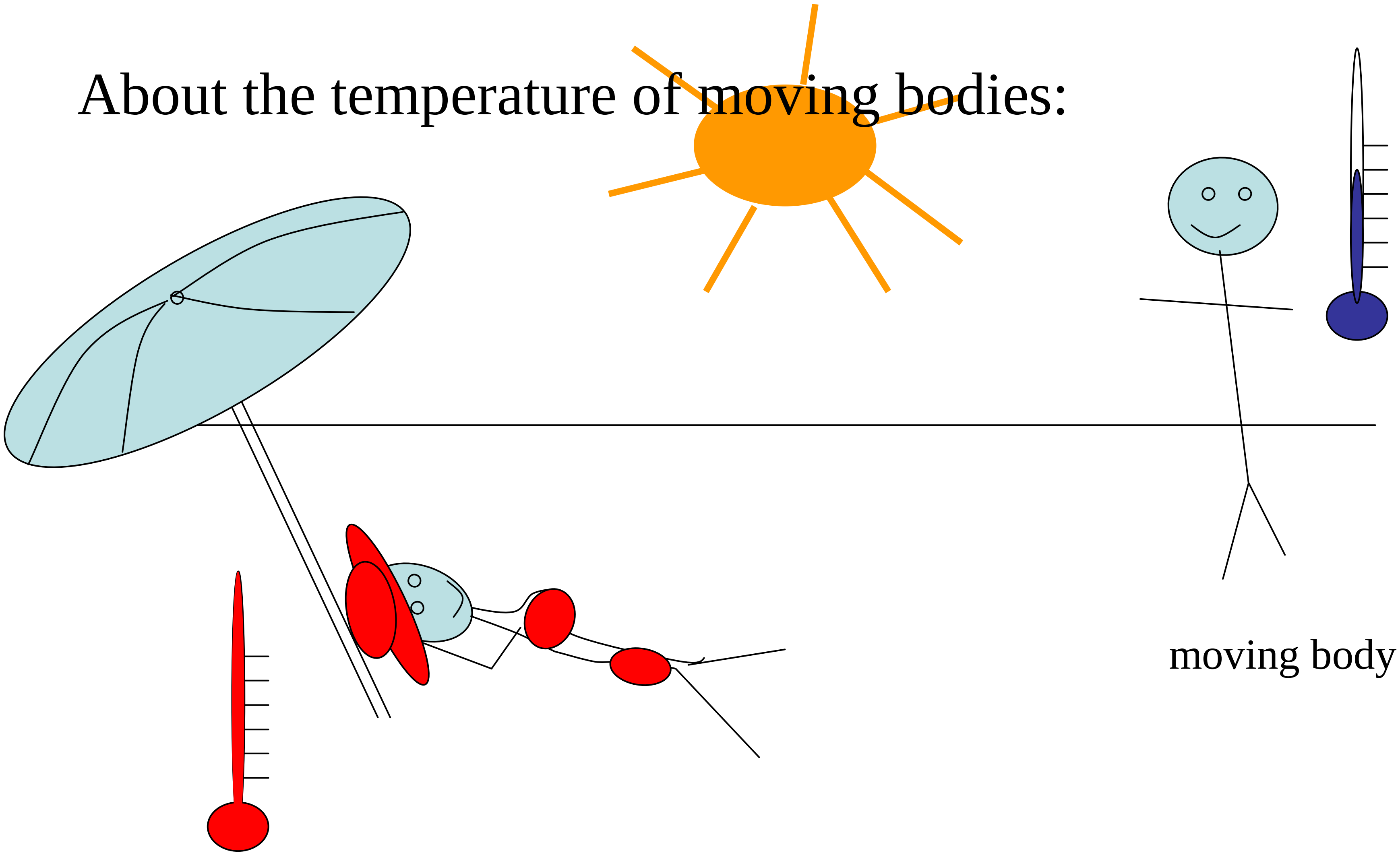
Conditions: thermodynamic stability, nothing more.

(Ván: arXiv:0811.0257)

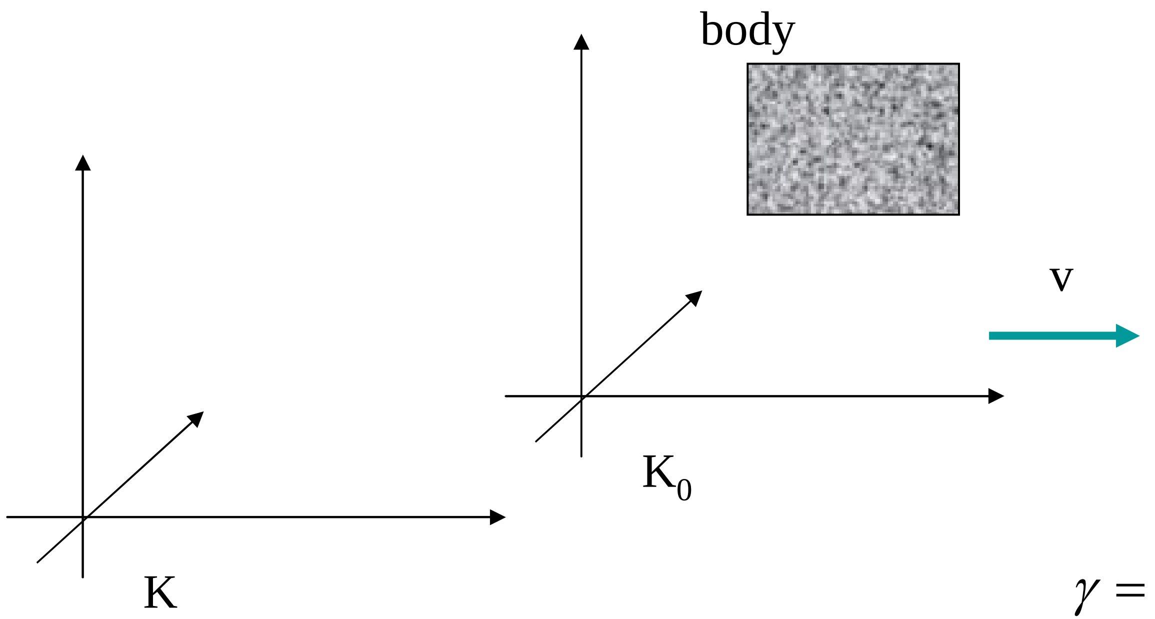
About the temperature of moving bodies:



About the temperature of moving bodies:



About the temperature of moving bodies:



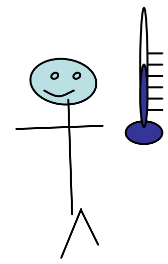
$$\gamma = \frac{1}{\sqrt{1-v^2}}$$

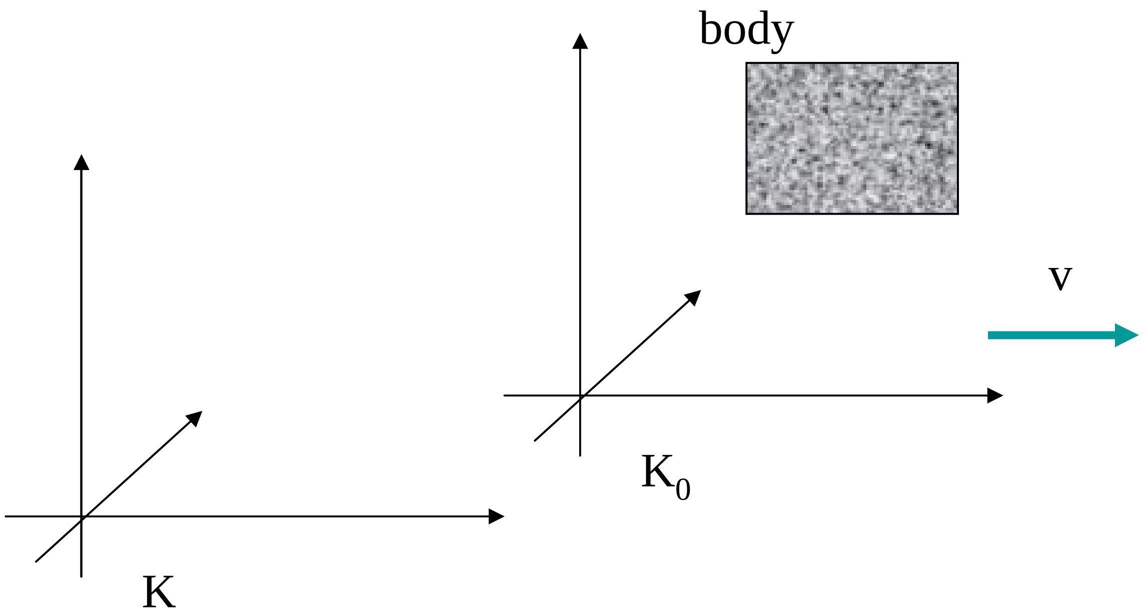
$$dE - \mathbf{v}d\mathbf{G} = TdS - pdV$$

translational work

<u>Einstein-Planck:</u> entropy is vector, energy + work is scalar

$$d\varepsilon = Tds, \quad s = \gamma s_0, \quad \varepsilon = \varepsilon_0 \quad \Rightarrow \quad \boxed{T = \gamma^{-1} T_0}$$



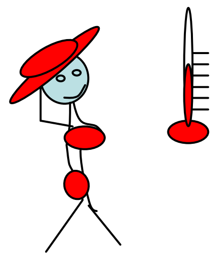


$$dE = TdS - pdV \qquad \gamma = \frac{1}{\sqrt{1-v^2}}$$

<u>Ott - hydro:</u> entropy is vector, energy-pressure are from a tensor

$$de = Tds, \quad s = \gamma s_0, \quad e = \gamma^2 e_0 \Rightarrow \boxed{T = \gamma T_0}$$

<u>Landsberg:</u> $\boxed{T = T_0}$



Thermostatics:

$$de - \frac{q^a}{e} dq_a = Tds + \mu dn \;\Leftrightarrow\; s\left(\sqrt{e^2 + q^a q_a}, n\right) = \hat{s}\left(e, q^a, n\right)$$

Temperatures and other intensives are doubled:

$$\frac{\partial s}{\partial \varepsilon} = \frac{1}{\Theta}; \quad \frac{\partial \hat{s}}{\partial e} = \frac{1}{T} \quad \Rightarrow \quad eT = \varepsilon\Theta, \;\; e\mu = \varepsilon \mathrm{M}$$

Different roles:

Equations of state: Θ, M
Constitutive functions: T, μ

$$q^a = -\lambda \Delta^{ac}\left(\partial_c T + T\dot{u}_c + T\frac{\dot{q}^a}{e}\right)$$

$$d\varepsilon = d\sqrt{e^2 - q^2} = \frac{ede - q^a dq_a}{\varepsilon} = \frac{e}{\varepsilon}\left(de - \frac{q^a}{\varepsilon} dq^a \right) = \theta ds + M dn$$

$$\boldsymbol{\varepsilon = \sqrt{\varepsilon^a \varepsilon_a}}$$

$\boldsymbol{\varepsilon^a = -u_b T^{ba}}$ **energy(-momentum) vector**

$$s = \gamma s_0, \quad \varepsilon = \gamma \varepsilon_0, \quad \Rightarrow \quad \theta = \theta_0$$

Landsberg

$$T = \frac{\theta \varepsilon}{e}, \quad e = \gamma^2 e_0 \quad \Rightarrow \quad T = \gamma^{-1} T_0$$

Einstein-Planck

non-dissipative $\Rightarrow$ $\boldsymbol{T = \gamma T_0}$ **Ott**

Summary

– Extended theories are not ultimate.

– energy ≠ internal energy
→ generic stability without extra conditions

– hyperbolic(-like) extensions, generalized Bjorken solutions, reheating conditions, etc…

Bíró, Molnár andVán: PRC, (2008), **78**, 014909 (arXiv:0805.1061)

– different temperatures in Fourier-law (equilibration) and in EOS out of local equilibrium
→ temperature of moving bodies - interpretation

Thank you for your attention!