Classical and quantum parts in Madelung variables

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1. Non-relativistic quantum mechanics
2. Special relativistic quantum mechanics
3. General relativistic quantum mechanics
4. Conclusions

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Symmetries, relativity and quantum mechanics

Wigner Jenő Pál around 1939:

ON UNITARY REPRESENTATIONS OF THE INHOMOGENEOUS LORENTZ GROUP*

By E. WIGNER

(Received December 22, 1937)

1939, …, 1990.

Space-time geometry of relativistic particles

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<u>Conformal symmetry:</u> a stepson in the family of spacetime symmetries.
(geometry, Weyl, …, gravitation and string theories.)

Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi$$

Madelung transformation

$$\psi = R e^{\frac{i}{\hbar}\alpha}$$

Classical action and momentum:

$$\frac{\partial \alpha}{\partial t} = -E, \qquad \nabla\alpha = P$$

$$\frac{\partial \psi}{\partial t} = \left(\frac{1}{R}\frac{\partial R}{\partial t} - \frac{i}{\hbar}E\right)\psi, \qquad \nabla\psi = \left(\frac{1}{R}\nabla R + \frac{i}{\hbar}P\right)\psi$$

Quantum content of quantum mechanics

Real part

$$E = V + \frac{P^2}{2m} - \frac{\hbar^2}{2m}\frac{\Delta R}{R}$$

Bohm potential

Imaginary part

$$m\frac{\partial R^2}{\partial t} + \nabla\left(R^2 P\right) = 0$$

continuity

Formal hydrodynamics:

$$P = mv\,,\; R^2 = \rho$$

energy equation (Bernoulli) – momentum balance (Korteweg)

Variational principle of Schrödinger

Multiplicative separation*:

Action–density variables – Hamilton-Jacobi correction:

$$\Sigma_{Schrödinger} = \int \left(\frac{\partial S}{\partial t} + \frac{|\nabla S|^2}{2m} + V \right) |\psi|^2 \, dx^3 \, dt$$

$$S = \frac{\hbar}{i} \ln(\psi)$$

Complex - mixture

Wave function variable – eikonal transformation:

$$\Sigma_{\psi} = \int \left(\frac{\hbar}{i} \bar{\psi} \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m} \nabla \bar{\psi} \cdot \nabla \psi + V \bar{\psi} \psi \right) dx^3 \, dt$$

$$S = \alpha - i \hbar \ln(R)$$

Additive separation:

Classical action-density variables – Madelung transformation:

$$\psi = R \, e^{\frac{i}{\hbar} \alpha}$$

$$\Sigma_{hydro} = \int \left[\left(\frac{\partial \alpha}{\partial t} + \frac{(\nabla \alpha)^2}{2m} + V \right) R^2 + \frac{\hbar^2}{2m} (\nabla R)^2 \right] dx^3 \, dt$$

Schrödinger, E., Zeitschrift für Physik, 1926.

Quantum – classical

$$\Sigma = \int \left[R^2 \left(\frac{\partial \alpha}{\partial t} + \frac{(\nabla \alpha)^2}{2m} + V \right) + \frac{\hbar^2}{2m} (\nabla R)^2 \right] dx^3 \, dt$$

Structure:

$$\Sigma = \int \left[R^2 \left(\begin{matrix} classical \\ Hamilton - Jacobi \end{matrix} \right) + \hbar^2 \left(\begin{matrix} quantum \\ kinetic \end{matrix} \right) \right] dx^3 \, dt$$

Klein-Gordon Lagrange density

$$L = \frac{1}{2}\partial_a \bar{\psi}\partial^a \psi - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2 \psi\bar{\psi} \qquad \left(\Sigma_{KG} = \int L\, d^4 x\right)$$

Here $diag\, \eta_{ab} = (1, -1, -1, -1)$, and *[L] = energy density/c = mc/L³*.

Madelung transformation:

$$\psi = \frac{\hbar}{\sqrt{mc}} R\, e^{\frac{i}{\hbar}\alpha}$$

$$\left(\frac{mc}{\hbar}\right)^2 \psi\bar{\psi} = mc\, R^2$$ → R^2 is a number density.

Normalization: $\int R^2 d^3 x = 2$

Transformed Klein-Gordon

Wave function:

$$L = \frac{1}{2}\partial_a \bar{\psi}\partial^a \psi - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2 \psi\bar{\psi}$$

$$\psi = \frac{\hbar}{\sqrt{mc}} R e^{\frac{i}{\hbar}\alpha}$$

Classical action – density:

$$L = \frac{\hbar^2}{2\mathrm{mc}}\left(\partial_a R \partial^a R + \frac{R^2}{\hbar^2}\left(\partial_a \alpha \partial^a \alpha - (mc)^2\right)\right) =$$

$$= \frac{mc}{2} R^2 \left(u_a u^a - 1\right) + \frac{\hbar^2}{2\mathrm{mc}} \partial_a R \partial^a R$$

Classical – quantum: $P_a = \partial_a \alpha\,,\quad u^a = \frac{P^a}{mc}$

Variations: U(1) Noether charge and mass shell

$$L=\frac{1}{2}\partial_a\bar{\psi}\partial^a\psi-\frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\psi\bar{\psi} \;=\; \frac{\hbar^2}{2\mathrm{mc}}\partial_a R\,\partial^a R+\frac{R^2}{2\mathrm{mc}}\left(\partial_a\alpha\,\partial^a\alpha-(mc)^2\right)$$

$$\delta_\alpha\Sigma=-\partial_a\left(\frac{R^2}{mc}\partial^a\alpha\right)=-\partial_a\left(R^2u^a\right)=\partial_a J^a=0$$

U(1) Noether charge

$$\delta_R\Sigma=-\frac{\hbar^2}{mc}\Box R+\frac{R}{mc}\left(P^aP_a-(mc)^2\right)=0$$

Off mass shell:

$$P^aP_a-(mc)^2 \;=\; \hbar^2\frac{\Box R}{R}$$

Metric view:

$$g_{ab}u^au^b \;=\; 1+\left(\frac{\hbar}{mc}\right)^2\frac{\Box R}{R}$$

Compton wavelength scaled and locally Lorentzian *spacetime metric.*

Energy-momentum tensor I – textbook construction

$$L=\frac{1}{2}\partial_a\bar{\psi}\partial^a\psi-\frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\psi\bar{\psi} = \frac{\hbar^2}{2\mathrm{mc}}\partial_a R\partial^a R+\frac{R^2}{2\mathrm{mc}}\left(\partial_a\alpha\partial^a\alpha-(mc)^2\right)$$

Noether:

$$T^{ab}_{KG}=\frac{1}{2}\left(\partial^a\bar{\psi}\partial^b\psi+\partial^a\psi\partial^b\bar{\psi}\right)-L\,g^{ab}=$$

$$=mc\,R^2u^a u^b+\frac{\hbar^2}{2\mathrm{mc}}\left(2\partial^a R\partial^b R-g^{ab}\left(\partial^c R\partial_c R+\partial^c_c R\right)\right)$$

dust + quantum

Energy-momentum tensor II – hydro construction

Fluid construction:

$$\frac{R^2}{2}\partial^b\left(u^a u_a - 1 - L_C^2\frac{\Box R}{R}\right) = \partial_a T^{ba}$$

Potential flow:

$$\partial^b u^a = \frac{1}{mc}\partial^{ab}\alpha = \partial^a u^b \qquad R^2 u_a \partial^b u^a = \partial^a\left(R^2 u_a u^b\right) - u^b\partial^a\left(R^2 u_a\right)$$

Fluid (pressure – Takabayashi) – particle (potential – Bohm):

$$R^2\partial^b\left(\frac{\Box R}{R}\right) = \partial^a\left(R\partial_{ab}R - \partial_a R\partial_b R\right)$$

$$T_{BT}^{ab} = mc R^2 u^a u^b - \frac{\hbar^2}{2\mathrm{mc}}\left(R\partial^{ab}R - \partial^a R\partial^b R\right)$$

dust + quantum

Energy-momentum tensor III

The difference of Klein-Gordon and Bohm-Takabayashi e-m tensors

$$T^{ab}_{diff} = T^{ab}_{KG} - T^{ab}_{BT} = \ldots = \frac{\hbar^2}{4\mathrm{mc}}\left(g^{ab}\Box - \partial^a \partial^b\right) R^2$$

Callan-Coleman-Jackiw term

Linear combination:

$$T^{ab} = T^{ab}_{BT} + \mu\, T^{ab}_{diff}$$

The trace:

$$T^a_a = \left[1 - L_C^2 \frac{1-3\mu}{4} \Box\right]\left(mc\, R^2\right)$$

μ=0	Bohm-Takabayashi
μ=1	Klein-Gordon
μ=1/3	*classical:* dust/*quantum:* traceless

Energy-momentum tensor: rescaled

$$R = \frac{e^{\sigma}}{\sqrt{V}} \qquad \left(\psi = \frac{\hbar}{\sqrt{mcV}} e^{\sigma + \frac{i}{\hbar}\alpha} \right)$$

V is a constant characteristic volume.

$$T^a_b = \frac{mc}{V} e^{2\sigma} u^a u_b + \frac{\hbar^2}{2mcV}\left(2\mu \partial^a \sigma \partial_b \sigma + (\mu - 1)\partial^a_b - \mu \delta^a_b (2\partial^c \sigma \partial_c \sigma + \Box \sigma)\sigma\right)$$

reminds to a dilaton

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$$+\frac{\hbar^2}{2\mathrm{mcV}}\left(2\mu\partial^a\sigma\partial_b\sigma+(\mu-1)\partial^a_b-\mu\delta^a_b(2\partial^c\sigma\partial_c\sigma+\Box\sigma)\sigma\right)$$

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Idea:

quantum energy-momentum – conformally curved spacetime

$$g_{ab} = e^{2s}\eta_{ab}$$

$$\hat{G}^{ab}_{curved} = \hat{T}^{ab}_{classical} \qquad G^{ab}_{plane} = T^{ab}_{quantum}$$

transformed — original

Remark:

scalar-tensor theories, Jordan vs. Einstein frames

Einstein equations

$$\hat{G}^a_b + 2\partial^a s\,\partial_b s - 2\partial^a_b s - \delta^a_b\left(2\partial^c s\,\partial_c s + \Box s + \Lambda\right) =$$

$$= \frac{8\pi G}{c^3} e^{-2\sigma} T^a_b = \qquad\qquad L_s = \frac{Gm}{c^2}$$

$$= \frac{8\pi L_s}{V} u^a u_b + \frac{4\pi L_s L_C^2}{V}\left(2\mu\,\partial^a\sigma\,\partial_b\sigma + (\mu - 1)\,\partial^a_b\sigma - \mu\,\delta^a_b\left(2\partial^c\sigma\,\partial_c\sigma + \Box\sigma\right)\right)$$

Solution: $\sigma = s$

$$V = \frac{4\pi}{3} L_s L_C^2 = \frac{4\pi}{3} L_s^3 \left(\frac{M_P}{m}\right)^4$$ scaled Planck volume

$$\mu = \frac{1}{3}$$ conformal quantum part

$$\Lambda = -3\left(\Box\sigma + \partial_c\sigma\,\partial^c\sigma\right) = -3\frac{\Box R}{R}$$

Quantum Einstein equation

$$G^a_b - 3\frac{\Box R}{R}\delta^a_b = mc\, R^2 u^a u_b$$

Off mass shell cosmological term:

$$\Lambda = 3\left((mc)^2 - P_a P^a\right) = 3(mc)^2\left(1 - u_a u^a\right)$$

Applications and ramifications

Cosmological constant (naturalness)

Thermodynamic background

beyond equilibrium, Fisher entropy, Korteweg fluids, ...

Hydrodynamics in field theory level

Jackiw et al., Kovtun, Dubovsky et al., ...

Speculation about naturalness

Particle in a Coulomb potential

$$\Lambda_{onium}^{nonrel} = -3\frac{\Box|\psi|}{|\psi|} = \frac{3}{A^2} - \frac{6}{Ar}$$

Here $A = L_C/\alpha$ is the Bohr radius.

$$\langle\Lambda\rangle = \frac{\int \Lambda(x)|\psi|^2 d^3x}{\int |\psi|^2 d^3x} = \frac{\sum_i \langle n_i\rangle V(A_i)\frac{3}{A_i^2}}{\sum_i \langle n_i\rangle V(A_i)} \qquad V(A_i) \sim A_i^3$$

$\langle n_i\rangle \approx const.$ maximal A_i dominates

$V\langle n_i\rangle \approx const.$ minimal A_i dominates

Coupled Majorana neutrinos – *neutrinonium*. Dark matter and dark energy.

Thank you for

your attention!

"And they said one to another: Go to, let us build us a tower, whose top may reach unto heaven; and let us make us a name. And the Lord said: Go to, let us go down, and there confound their language, that they may not understand one another's speech."

Genesis 11: 3–7

Backup slides…

Field/continuum/fluid OR/AND particle?

Non-relativistic

Momentum balance

$$\underline{\rho \dot{v}^i + \partial_j P^{ij}_{quantum}} = \rho \underline{\left(\dot{v}^i + \partial^i U_{quantum}\right)}$$

$$\partial_j P^{ij}_{quantum}(\rho, \nabla\rho, \nabla^2\rho) = \rho\, \partial^i U_{quantum}(\rho, \nabla\rho, \nabla^2\rho)$$

Korteweg fluid

Relativistic

Energy-momentum balance

$$\partial_b T^{ab} = \partial_b\left(\rho u^a u^b + P^{ab}\right) = \underline{\rho \dot{u}^a + \partial_\nu P^{ab}_{quantum}} = \rho \underline{\left(\dot{u}^a - \partial^a U_{quantum}\right)}$$

E.g.

$$\partial^a\left(R\,\partial_{ab} R - \partial_a R\,\partial_b R\right) = R^2 \partial^b\left(\frac{\Box R}{R}\right)$$

Related ideas:

– Scalar-tensor theories (Brans-Dicke, quintessence, etc...)
Jordan and Einstein frames – physical equivalence

– Delphenic, Sohai-Soahi, Weyl geometry

– Sakharov – constitutive Einstein equation

– Changing mass ideas by Bohmian term.

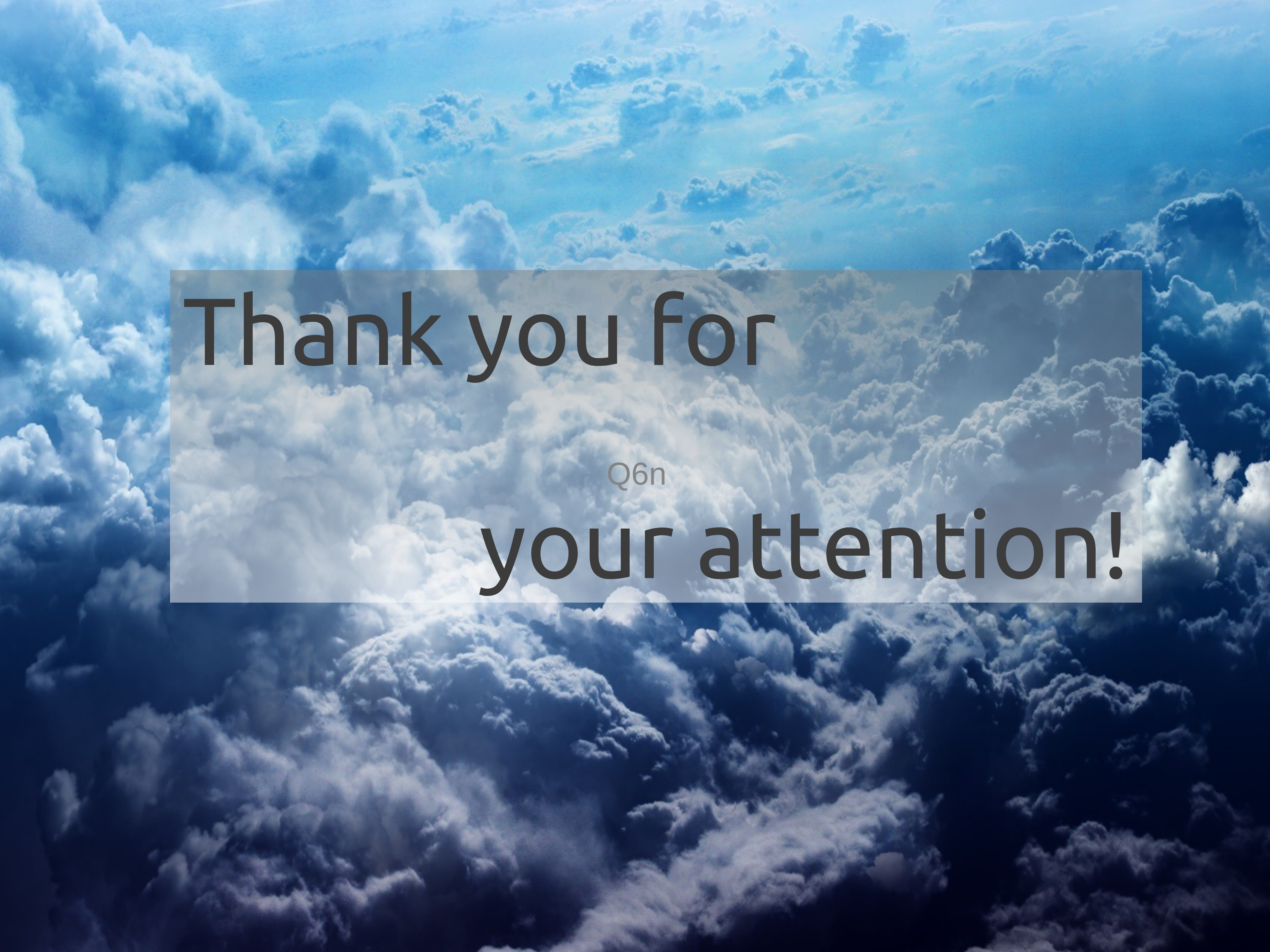
„If we have to go on with these damned quantum jumps, then I'm sorry that I ever got involved.”

E. Schrödinger

”I would like to make a confession which may seem immoral: I do not believe in Hilbert space anymore.”

J. von Neumann

(in a letter to G.D. Birkhoff, 1935)


Thank you for
Q6n
your attention!